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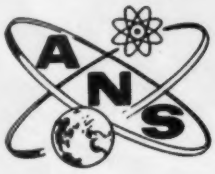

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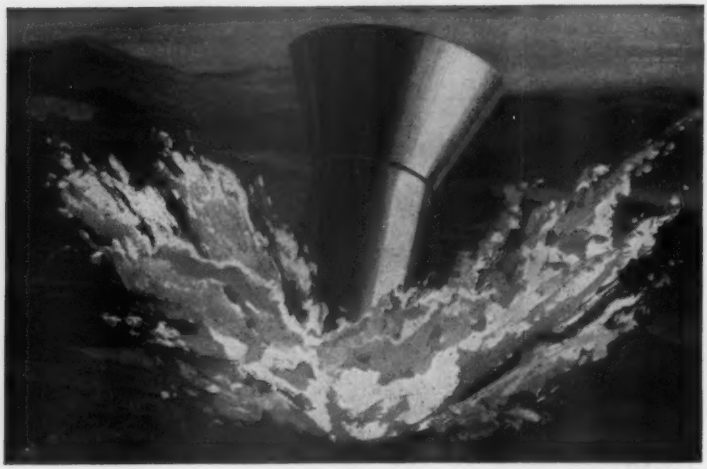

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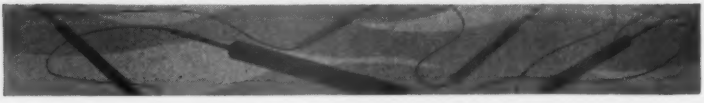

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PREDICTION OF LOCAL SKIN-FRICTION COEFFICIENT BY TRANSFORMATION OF THE COMPRESSIBLE BOUNDARY LAYER WITH HEAT TRANSFER

BY

P. K. CHANG [1] AND H. B. NOTTAGE [2]

ABSTRACT

This paper presents a transformation between compressible and incompressible two-dimensional and axisymmetric boundary-layer flows. The transformation may be used to predict local skin-friction coefficients for laminar and for turbulent flows, in zero as well as in favorable pressure gradient, and also for conditions with surface heat transfer through the use of the reference temperature. Application is limited, in the present version, to Prandtl number in the order of unity. The final form of the transformation involves a factor to be determined by comparison with experiment, and in the present case this is done for a particular range of data on air. Accordingly, quantitative application of the results will have greatest assurance for air over this same range, pending further data on other fields and ranges of flow conditions.

Sample computations illustrate the application of the transformation to (1) turbulent flow of air along an insulated flat plate with free stream Mach number of 3.7, and (2) laminar boundary layer of air along a sphere with free stream Mach number of 2.855. These cases demonstrate good agreement between predicted and measured skin-friction coefficients as affected by aerodynamic heating.

INTRODUCTION

Coordinate transformation, whereby corresponding points in compressible and incompressible boundary-layer flows are related, are very useful in reducing the inherent complexity of the analytical treatment of compressible flows. Stewartson (1)[3] and Illingworth (2) independently derived the transformation for laminar flow along insulated sur-

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[3] The boldface numbers in parentheses refer to the references appended to this paper.

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faces. They assumed a Prandtl number of unity and equality between the stagnation and total temperatures.

Subsequently, Englert (3) transformed the equation of Truckenbrodt (4), which is useful for the computation of momentum thickness. In this he applied Stewartson's laminar flow transformation and assumptions to turbulent as well as to laminar flow. Mager (5) formulated a transformation for turbulent flow, with unity Prandtl number and adiabatic wall, on the basis of including the mean fluctuating velocity components and apparent turbulent shear in the boundary-layer equations. Mager's transformation reduces to Stewartson's if the absolute viscosity varies proportionally with absolute temperature.

The velocity profile in a turbulent boundary layer is conveniently represented by a power law. Englert's and Mager's transformations for turbulent flow, however, are independent of the power-law exponent. Furthermore, their results indicate that the relation[4] $c_f/c_{fi} = T_e/T_0$ is applicable for both laminar and turbulent flows. Cohen and Reshotko (6) applied the method of Thwaites (7) and modified Stewartson's transformation to include the effect of heat transfer. Also, by applying the procedure of Tifford and Chu (8), Cohen and Reshotko extended the heat-transfer treatment to include Prandtl number other than unity.

Hill (9) derived a generalized transformation which is applicable to both laminar and turbulent flows, and which involves the velocity-profile parameter. To account for heat transfer, Hill applied the concept of Eckert's reference temperature (10) whereby, if the fluid properties are evaluated at this temperature, the cases of compressible flow with surface heat transfer can be predicted quite well.

A direct transformation between compressible and incompressible flow is established in this paper. The y-coordinate analysis is obtained by introducing the same stream function at corresponding points for the incompressible and compressible flows. The x-coordinate analysis is obtained by an improvement of Hill's transformation derived from momentum relationships. This improvement is expressed as a multiplicative correction factor which is determined by experimental data. The data employed for this purpose cover Mach numbers ranging from 2 to 5, with ratios of reference temperature to free stream static temperature for air from 1.58 to 4.30. Further evidence on the applicability of this improvement is established by comparing measured and computed skin-friction coefficients in the following two cases:

1. Turbulent flow along an insulated flat plate with free-stream Mach number of 3.7; and
2. Laminar boundary layer along a heat-conducting sphere with free-stream Mach number of 2.855.

The favorable outcome from these comparisons serves to recommend

[4] The Appendix lists the symbols used in this paper.

the general use of the transformation for design predictions of skin friction, within the range of validation, for both laminar and turbulent flows of air in zero and favorable pressure gradients under the influence of aerodynamic heating.

ANALYSIS

The derivations are given below for both the x-transformation and the y-transformation. These results relate the coordinates in the compressible and incompressible boundary-layer flows at which the same states prevail. This is, where $x = x_i$, $y = y_i$, then the velocities, boundary layer, geometry, and local skin-friction coefficient are the same for the two types of flow.

TRANSFORMATION OF THE y-COORDINATE

The principle of the transformation requires that the stream functions be the same for both flows; that is, $\psi = \psi_i$ at corresponding coordinates (x, y) and (x_i, y_i). The stream functions for the two cases, satisfying the respective continuity equations, are:

$$\psi_y = \frac{\rho u}{\rho_0}, \qquad \psi_x = -\frac{\rho v}{\rho_0},$$

$$\psi_{y_i} = u_i, \qquad \psi_{x_i} = -v_i.$$

For $\psi = \psi_i$, it follows that

$$y_i = \int_0^{\psi} \frac{\partial \psi}{u_i} = \frac{\rho_e u_e}{\rho_0 u_0} \int_0^y \frac{u_{e_i}}{u_i} \frac{\rho u}{\rho_e u_e} dy.$$

Experiment has shown that variations in shear stress and unit convective conductance have only a slight effect upon the velocity and temperature profiles in a boundary layer (11), and as a result of compressible boundary-layer tests extending up into the low hypersonic range, Mach number of 7, Gruenwald and Lobb (12) showed that

$$\frac{u}{u_e} = \frac{u_i}{u_{e_i}} = \left(\frac{y}{\delta}\right)^{1/n} = \left(\frac{y}{\delta}\right)_i^{1/n_i}.$$

The transformation of the y-coordinate follows directly as:

$$y_i = \frac{\rho_e u_e}{\rho_0 u_{e_i}} \int_0^y \left(\frac{\rho}{\rho_e}\right) dy = \left(\frac{T_e}{T_0}\right)^{(\gamma+1)/2(\gamma-1)} \int_0^y \left(\frac{\rho}{\rho_0}\right) dy. \qquad (1)$$

This is the transformation derived by Illingworth and by Stewartson for laminar flows, by Cohen and Reshotko for laminar flow, by Mager for turbulent flow and by Hill and by Englert for both laminar and turbulent flows.

Since

$$\psi_i = \int u_i \partial y_i \qquad \text{and} \qquad \psi = \int \left(\frac{\rho}{\rho_0}\right) u dy,$$

the condition for equal stream functions yields

$$\int u_i \partial y_i = \int \left(\frac{\rho}{\rho_0}\right) u dy.$$

Using Eq. 1,

$$u_i = u(a_e/a_0). \tag{2}$$

TRANSFORMATION OF THE x-COORDINATE

The transformation of the x-coordinate is formulated by identifying equivalent forms of the momentum equations for incompressible and compressible flow, and by introducing an experimental factor obtained from skin-friction data. To establish the desired identity, relationships are needed between θ and H and θ_i and H_i, respectively. By definition

$$\theta_i = \int_0^\infty \frac{u_i}{u_{ei}} \left(1 - \frac{u_i}{u_{ei}}\right) dy_i$$

and

$$\theta = \int_0^\infty \frac{\rho u}{\rho_e u_e} \left(1 - \frac{u}{u_e}\right) dy.$$

Then, using Eq. 1

$$\theta_i = (T_e/T_0)^{(\gamma+1)/2(\gamma-1)} \cdot \theta. \tag{3}$$

The displacement thickness of the boundary layer is by definition,

$$\delta^* = \int_0^\infty \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy.$$

The ratio of density may be given by the relation (3),

$$\frac{\rho_e}{\rho} = 1 + \frac{\gamma - 1}{2} \mathrm{M}^2 \left(1 - \frac{u_i^2}{u_{ei}^2}\right).$$

This equation holds for an insulated wall with Prandtl number of unity.

Introducing the y-coordinate transformation, the displacement thickness becomes

$$\delta^* = \left(\frac{T_0}{T_e}\right)^{(\gamma+1)/(\gamma-1)} \left[\delta_i^* + \frac{\gamma - 1}{2} \mathrm{M}^2(\delta_i^* + \theta_i)\right].$$

The boundary-layer shape parameters for compressible and incompressible flow then are related by the expression

$$H = \frac{\delta^*}{\theta} = H_i + \frac{\gamma - 1}{2} \mathrm{M}^2(H_i + 1)$$

or

$$H + 1 = T_0/T_e(H_i + 1). \tag{4}$$

Culick and Hill (13) found by experiment that Eq. 4 holds for an adiabatic surface with Pr = 1. Cohen and Reshotko (6) used this equation for the non-insulated surface heat transfer. This assumption is compatible with taking the velocity-profile parameter n to be independent of compressibility effects and using the reference temperature (10) to account for the influence of heat transfer. Since the present theory introduces a factor to be determined by experiment, cumulative consequences of these approximations will influence the consistency or inconsistency of this factor, and if the factor is consistent, then the procedure has a useful outcome.

Using Eqs. 3 and 4 Hill (9) showed how to relate dx_i and dx from the momentum equations of the boundary layers. The momentum equation for compressible flow along a body of revolution is

$$\frac{d\theta}{dx} + \theta\left[(H + 2 - \mathrm{M}^2)\frac{1}{u_e}\frac{du_e}{dx} + \frac{1}{r}\frac{dr}{dx}\right] = \frac{\tau_w}{\rho_e u_e^2}.$$

Introducing Eqs. 3 and 4, this becomes

$$\frac{d\theta_i}{dx_i} + \theta_i\left[(H_i + 2)\frac{1}{u_{ei}}\frac{du_{ei}}{dx_i} + \frac{1}{r}\frac{dr}{dx_i}\right] = \frac{\tau_w}{\rho_e u_e^2}\left(\frac{T_e}{T_0}\right)^{(\gamma+1)/2(\gamma-1)}\frac{dx}{dx_i}.$$

The corresponding momentum equation for incompressible flow is

$$\frac{d\theta_i}{dx_i} + \theta_i\left[(H_i + 2)\frac{1}{u_{ei}}\frac{du_{ei}}{dx_i} + \frac{1}{r}\frac{dr}{dx_i}\right] = \left(\frac{\tau_w}{\rho_e u_e^2}\right)_i.$$

The last two equations are the same if

$$\frac{dx_i}{dx} = \left(\frac{c_f}{c_{fi}}\right)\left(\frac{T_e}{T_0}\right)^{(\gamma+1)/2(\gamma-1)}. \tag{5}$$

This result is one preliminary form of the x-coordinate transformation.

USE OF SKIN FRICTION COEFFICIENT DATA TO ESTABLISH THE x-COORDINATE TRANSFORMATION

Availability of c_f/c_{fi} data for a flat plate allows Eq. 5 to be reduced to a directly useful form. The skin-friction coefficient for a flat plate,

based on Re_θ, is for incompressible flow (9)

$$\frac{c_{f_i}}{2} = b\left(\frac{\rho_e u_e \theta}{\mu_e}\right)_i^{-2/(n+1)}$$

where b is a factor which is a function of n only.

The similar expression for compressible flow, with n the same and b^* a function of n only, is

$$\frac{c_f}{2} = b^*\left(\frac{\rho_e u_e \theta}{\mu_e}\right)^{-2/(n+1)}.$$

Then, for comparisons under similar conditions, namely, the same θ, u_e, and n:

$$\frac{c_f}{c_{f_i}} = \frac{b^*}{b}\left(\frac{\mu^*}{\mu_e}\right)^{2/(n+1)}\left(\frac{\rho^*}{\rho_e}\right)$$

where superscript * applies to the compressible boundary layer wherein properties are to be evaluated at the reference temperature, which includes the effect of heat transfer.

Let the ratio $b^*/b = C$, which is a function n only; then

$$\frac{c_f}{c_{f_i}} = C\left(\frac{\mu^*}{\mu_e}\right)^{2/(n+1)}\frac{\rho^*}{\rho_e}. \tag{6}$$

The next step is to relate the fluid properties in Eq. 6 to the corresponding absolute temperature. In the particular range of $500°\mathrm{R} \leqslant T \leqslant 700°\mathrm{R}$, the dynamic viscosity μ is essentially proportional to T and the perfect gas law is applicable. Under these conditions, and these only, Eq. 6 becomes

$$c_f/c_{f_i} = C(T^*/T_e)^{(1-n)/(1+n)}. \tag{7}$$

The modified reference temperature (14) is given by Sommer and Short as follows:

$$\frac{T^*}{T_e} = 1 + 0.035\,\mathrm{M}^2 + 0.45\left(\frac{T_w}{T_e} - 1\right). \tag{8}$$

Korkegi (15) measured c_f on an insulated flat plate and evaluated c_f/c_{f_i} for the same Re_θ with $n = 5$. Using his data, the factor C in Eqs. 6 and 7 can be computed. Results are summarized in Table I.

The nature of the assumptions made in the theory and of the limited data available do not warrant any refinement beyond using the average $C = 1.06$. Further data are needed to illuminate this point, and, if available, the theory can be refined accordingly. Indeed,

TABLE I.—*Values of* C *Determined According to the Data of Korkegi* (15) *on an Insulated Flat Plate.*

$\mathrm{Re}_\theta = 4000$ $T_0 = 685°\mathrm{R}$, and $n = 5$

M	2	3	4	5
c_f/c_{f_i}	0.83	0.63	0.53	0.44
$\left(\frac{\mu^*}{\mu_e}\right)^{2/(1+n)} \frac{\rho^*}{\rho_e} = \left(\frac{T^*}{T_e}\right)^{(1-n)/(1+n)}$	0.778	0.62	0.4975	0.403
$C = \frac{b^*}{b}$	1.07	1.016	1.066	1.092

Average value of $C = 1.06$

it is hoped to extend the analysis along such lines in future experimental investigations. In the present, as a working basis within the range imposed, it is proposed that the x-coordinate transformation should be developed as follows:

μ and ρ of incompressible flow are evaluated at stagnation conditions of μ_0 and ρ_0. Therefore,

$$(\mathrm{Re}_\theta)_i/\mathrm{Re}_\theta = \mu_e/\mu_0$$

and, in general, $\mathrm{Re}_{\theta_i} \neq \mathrm{Re}_\theta$; for this condition, Eq. 6 reduces to

$$\frac{c_f}{c_{f_i}} = C \left(\frac{\mu^*}{\mu_0}\right)^{2/(n+1)} \frac{T_e}{T^*}. \tag{9}$$

Finally, the x-coordinate transformation becomes

$$\left.\begin{aligned} \frac{dx_i}{dx} &= 1.06 \left(\frac{\mu^*}{\mu_0}\right)^{2/(n+1)} \left(\frac{T_e}{T_0}\right)^{(\gamma+1)/2(\gamma-1)} \frac{T_e}{T^*} \\ \text{or} \qquad x_i &= 1.06 \int_0^x \left(\frac{T_e}{T_0}\right)^{(3\gamma-1)/2(\gamma-1)} \left(\frac{\mu^*}{\mu_0}\right)^{2/(n+1)} \frac{T_0}{T^*}\, dx \end{aligned}\right\}. \tag{10}$$

SAMPLE APPLICATIONS

Utility of the transformations is investigated by two sample applications for which measured skin-friction coefficients are available, namely, for turbulent flow along an insulated flat plate, and for a laminar boundary layer along a heat-conducting sphere.

1. Turbulent Flow along an Insulated Flat Plate

Coles (16) measured $c_f = 0.00162$ under the following conditions: $\mathrm{M} = 3.70$, $p_0 = .030$ mm Hg, $T_0 = 561°\mathrm{R}$, $x = 21.48$ in. from leading edge, $\mathrm{Re}_x = 3.54 \times 10^6$, $n = 5$, $H = \delta^*/\theta = 7.94$, and $r = (T_w - T_e)/(T_0 - T_e) = 0.896$.

To compute c_f, first compute x_i by transformation of x-coordinate, and compute c_{f_i} at x_i. Then compute c_f by means of transformation applied to skin-friction coefficient, and compare with measured c_f.

Use Eq. 8 and apply the approximation that μ varies proportionally with T. $\gamma = 1.4$, $T^*/T_e = 2.58$, $\mu^*/\mu_0 = T^*/T_0 = 0.69$, and $C = 1.06$. From Eq. 10,

$$x_i = 1.06 \int_0^x \left(\frac{T_e}{T_0}\right)^{(3\gamma-1)/2(\gamma-1)} \left(\frac{\mu^*}{\mu_0}\right)^{2/(n+1)} \frac{T_0}{T^*}\, dx$$

$$= 0.01235 \text{ ft.}$$

To predict c_{f_i}, a relation is needed involving Re_i and n; this is obtained as follows:

For an incompressible boundary layer, the "friction velocity" is

$$u^+ = \sqrt{\left(\frac{\tau_w}{\rho_e}\right)_i}$$

or

$$\left(\frac{\tau_w}{\rho_e u_e^2}\right)_i = \left(\frac{u^+}{u_e}\right)_i.$$

The velocity power law may be expressed as (17),

$$\left(\frac{u}{u_e}\right)_i = C_i \left(\frac{yu^+}{\nu}\right)_i^{1/n}$$

and

$$\left(\frac{u}{u_e}\right)_i = \left(\frac{y}{\delta}\right)_i^{1/n}.$$

Hence, the boundary-layer thickness is

$$\delta_i = \left(\frac{\nu_e}{C_i u_e}\right)_i \lambda_i^{(n+1)/2}$$

where

$$\lambda_i = \left(\frac{\tau_w}{\rho_e u_e^2}\right)_i.$$

The corresponding momentum thickness of boundary layer is

$$\theta_i = \frac{n}{(n+1)(n+2)} \left(\frac{\nu_e}{C_i u_e}\right)_i \left(\frac{1}{\lambda_i}\right)^{(n+1)/2}.$$

For a flat plate, the momentum equation reduces to

$$\left(\frac{d\theta}{dx}\right)_i = \frac{1}{\lambda_i}.$$

Hence

$$\left(\frac{u_e}{\nu_e}\right)_i \cdot dx_i = \frac{n}{2(n+2)C_i^n}\,\lambda_i^{(n+1)/2} d\lambda_i.$$

Integrating, C_i is related to $(\mathrm{Re}_x)_i$

$$(\mathrm{Re}_x)_i = \left\{\frac{n}{(n+1)(n+3)C_i^n}\right\}^{2/(n+3)} (\mathrm{Re}_x)_i^{-2/(n+3)}.$$

For $n = 5$, the value of C_i is found by an extrapolation (**18**). Values of n from 7 to 10 can be plotted in a function of C_i by a straight line. By extending this straight line to $n = 5$, $C_i = 6.9$ is determined.

Then finally

$$c_{fi} = \frac{0.0977}{(\mathrm{Re}_x)_i^{1/4}}.$$

For the computation of $(\mathrm{Re}_x)_i$ corresponding to compressible flow, the properties are evaluated at stagnation conditions. Thus,

$$u_{ei} = a_0 \mathrm{M} = 4300 \text{ ft/sec.}$$

$$\nu_{ei} = \nu_0 = \frac{\mu_0}{\rho_0} = 1.36 \times 10^{-4} \text{ ft}^2/\text{sec.}$$

$$(\mathrm{Re}_x)_i = \frac{u_{ei} x_i}{\nu_0} = 39.3 \times 10^4.$$

Hence

$$c_{fi} = \frac{0.0977}{(\mathrm{Re}_x)^{1/4}} = 0.0390.$$

Taking μ proportional to T with $T_0 = 561°\mathrm{R}$, from Eq. 8, and $n = 5$, Eq. 9 gives

$$c_f = c_{fi}(1.06)\left(\frac{\mu^*}{\mu_0}\right)^{2/(n+1)} \frac{T_e}{T^*} = 0.00142.$$

The data of Coles give $c_f = 0.00162$, which is some 12 per cent higher than the predicted magnitude. Considering the complicated nature of the physical situation and the simplifying postulates which were employed to formulate the relatively simple solution, the agreement is satisfactory.

2. Laminar Boundary Layer along a Heat-Conducting Sphere

The shear stress at a point on a brass sphere has been measured at the David Taylor Model Basin (**19**) in accordance with Fig. 1.

The measured shear stress at $\varphi = 80°$, $\tau_w = 0.00384$ psi. Assuming the Mach number on a sphere (**20**) to vary as $\frac{3}{2}\varphi$, when $\varphi = 80°$, then M = 2.1. Also, $T_e = 287.5°$R, $u_e = 1750$ ft/sec, and $\rho_e = 0.000174$ slug/ft^3; thus the measured skin-friction coefficient is

$$c_f = \tau_w/(\tfrac{1}{2}\rho_e u_e^2) = 0.00208.$$

Prediction of c_{f_i} follows the transformation as expressed in Eq. 9

$$\frac{c_f}{c_{f_i}} = C\left(\frac{\mu^*}{\mu_0}\right)^{2/(n+1)}\left(\frac{T_e}{T^*}\right).$$

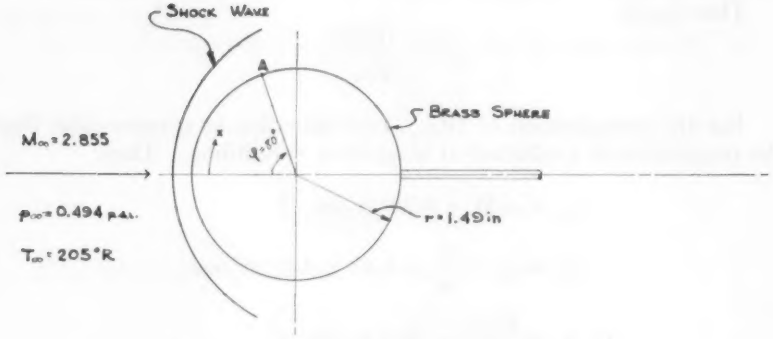

FIG. 1. A sphere submerged into compressible flow.

Assuming as before that μ is proportional to T, and $C = 1.06$, then for laminar flow $n = 1$, $c_f = 0.531 c_{f_i}$.

The distance x_i is calculated by Eq. 10

$$x_i = 1.06\int_0^x \left(\frac{T_e}{T_0}\right)^{(3\gamma-1)/2(\gamma-1)}\left(\frac{\mu^*}{\mu_0}\right)^{2/(n+1)}\frac{T_0}{T^*}\,dx$$

with $M \approx \frac{3}{2}\varphi$ and $\gamma = 1.4$,

$$x_i = 1.06r\int_0^{(80/180)\pi}\left(1+\frac{9}{20}\varphi^2\right)^{-4}d\varphi = 0.742r.$$

Since $r = 0.124$ ft, $x_i = 0.0920$ ft.

For a laminar boundary layer around a sphere (**21**)

$$\frac{c_{f_i}}{2} = \frac{1.32}{\left(\dfrac{u_e x}{\nu_e}\right)_i^{1/2}}.$$

Furthermore,

$$u_{ei} = \frac{a_0}{a_e} u_e = 2400 \text{ ft/sec}$$

and

$$\nu_0 = 4.65 \times 10^{-4} \text{ ft}^2/\text{sec}.$$

Therefore,

$$c_{f_i} = 0.00384.$$

Finally, the computed value at $\varphi = 80°$ is

$$c_f = 0.531 c_{f_i} = 0.00204.$$

Since the measured value in $c_f = 0.00208$, the transformation has served very well for this case of a laminar boundary layer in a favorable pressure gradient.

DISCUSSION

The x-coordinate transformation as detailed in this paper introduces the power-law parameter n for the boundary-layer velocity profile, the reference temperature T^* to account for heat transfer, an expedient assumption that μ is proportional to T for the range of application, and a factor C to be established by experiment to account for the influence of compressibility in the variation of c_f with Reynolds number based on momentum thickness. The variation of μ with T can be generalized to other temperature ranges by using a power law. The factor C is an interim empiricism upon which further work is to be done toward a more general basis of prediction. Extensions to other Prandtl numbers will be treated in the future.

For engineering purposes, the type of analysis presented here is useful as a prediction tool, and it is fairly easy to apply.

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APPENDIX

Notation

a = velocity of sound
C = constant
c_f = local skin-friction coefficient
H = shape parameter, $H = \delta^*/\theta$
M = local Mach number of free stream
n = parameter of boundary-layer velocity profile
Pr = Prandtl number
Re_x = Reynolds number based on distance x
Re_θ = Reynolds number based on momentum thickness
r = radius
T = absolute temperature
u = streamwise velocity within the boundary layer
u_e = streamwise velocity at the outer edge of boundary layer
u^+ = shear velocity
v = normal velocity to the surface of body
x = distance along the surface from leading edge or from stagnation point
y = normal distance to the surface
δ = thickness of boundary layer
δ^* = displacement thickness of boundary layer
γ = ratio of specific heats
θ = momentum thickness of boundary layer
ρ = density
μ = absolute viscosity
τ = shear stress
φ = angle measured from stagnation point

Subscripts

i = incompressible
e = free stream at the outer edge
w = wall
0 = stagnation

Superscript

* = values based on reference temperature

ON THE VALIDITY OF THE STATIC APPROXIMATION FOR THE EVALUATION OF THE TWO-NUCLEON POTENTIAL *

BY

MALCOLM C. YOUNGER [1] AND ABRAHAM KLEIN [1]

ABSTRACT

The perturbative two-pion exchange nuclear potential is computed in an approximation which does not neglect the possible large recoil of the nucleons upon emission or absorption of a virtual pion. Upon the assumption that initial and final momenta are nonrelativistic, the result is still a local potential, but one to which previous completely static approximations bear only a qualitative resemblance. The corrections are of order (μ/Mx) where μ and M are the pion and nucleon masses, respectively, and x is the nucleon separation in units of the pion Compton wavelength.

I. INTRODUCTION

The purpose of this work is to investigate the reliability of the static limit for evaluating the two-pion-exchange internucleon potential. The determination of the corrections to the static limit is based on the definition of the potential given by Klein and McCormick (1).[2] Therefore the static limit under consideration is the potential of Brueckner and Watson (2) insofar as rescattering corrections are omitted.

The static limit refers to the limit $M \to \infty$, where M is the nucleon mass. Hence, it is considered permissible to make expansions in powers of the momenta of all particles, whether in the initial, final, or intermediate state, insofar as these momenta are compared to the nucleon mass, M. Recently, this procedure has been questioned sharply by several authors (3, 4, 5), who have pointed out that, in fact, all such expansions may not be valid. These criticisms have been given within the framework of new or refurbished programs for the computation of the potential. The superiority of these programs compared to previous ones, though it may turn out to be real, has yet to be demonstrated in practice. For this reason, and also because the static potential proposed by Klein and McCormick is now being compared with experiment (6), we considered it worthwhile to carry out an investigation of its intrinsic accuracy.

In terms of the present formulation, expansion in powers $\mathbf{k}M^{-1}$, where $\mathbf{k}$ is the momentum of one of the virtual pions exchanged by the two nucleons, and therefore can be arbitrarily large, needs further study,

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[2] The boldface numbers in parentheses refer to the references appended to this paper.

and will be investigated in this work. On the other hand, expansion in powers of $\mathbf{p}M^{-1}$, where $\mathbf{p}$ is the relative momentum of the two nucleons, still will be credited. For small relative momenta, $p \ll M$, such an expansion should be permissible, since no integration is performed over $\mathbf{p}$. Hereinafter, the result of expanding the expression for the potential in powers of $\mathbf{p}M^{-1}$ will be called the adiabatic limit, to distinguish it from the more restrictive and questionable static limit.

There have, of course, been prior investigations of corrections to the static potentials (7), but none of these can be compared with the results of the present work, for the dual reasons that they have been based on older, physically less adequate formulations of the potential problem and that they have considered indiscriminately both non-static and *non-adiabatic* corrections. Nevertheless, in their numerical aspects these previous investigations have already demonstrated the possible importance of such corrections as we shall consider.

We turn finally to a description of the basic ideas of our work. As stated above, we continue to sanction an expansion in $\mathbf{p}M^{-1}$ and indeed we shall restrict ourselves to an account of the zero order term, though it would be simply a matter of persistence to obtain the higher order terms by the methods to be detailed. For the two-meson-exchange potential we obtain an integral of the form

$$\int d^3k_1 \int d^3k_2 f(\mathbf{k}_1, \mathbf{k}_2, M, \mu) e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}} \tag{1}$$

where $\mathbf{r}$ is the two nucleon separation, μ the pion mass, and $\mathbf{k}_1$, $\mathbf{k}_2$ the meson momenta. Thus for internucleon separations of the order of the pion Compton wavelength, $r_0 = \mu^{-1}$, or larger, the dominant contributions to the potential arise from values of $\mathbf{k}_1$ and $\mathbf{k}_2$ which satisfy $|\mathbf{k}_1 + \mathbf{k}_2| \lesssim r_0^{-1}$. Consequently it should be legitimate to expand the potential in powers of $|\mathbf{k}_1 + \mathbf{k}_2|M^{-1}$ if interest is restricted to large internucleon separations, $r \gtrsim r_0$. However, the static limit is reached by making expansions in powers of both $\mathbf{k}_1M^{-1}$ and $\mathbf{k}_2M^{-1}$ individually rather than in powers of $|\mathbf{k}_1 + \mathbf{k}_2|M^{-1}$ alone. In the present work we shall carry through the evaluation of (1) permitting the latter expansion only.

The actual quantity to be computed in the present work is the difference between the adiabatic and static limits for the two-pion-exchange potential, exclusive of rescattering corrections (1). Hereinafter this quantity will be called $\delta V(r)$. The omission thus far of a corresponding treatment of the pion-nucleon rescattering corrections, coupled with the substantial magnitudes of the corrections found, means that we have substantiated the importance of the effect investigated, but the theory must be extended before it can be confronted consistently with experiment in its non-static form.

Our results are summarized in Figs. 1–6. It is seen that in most

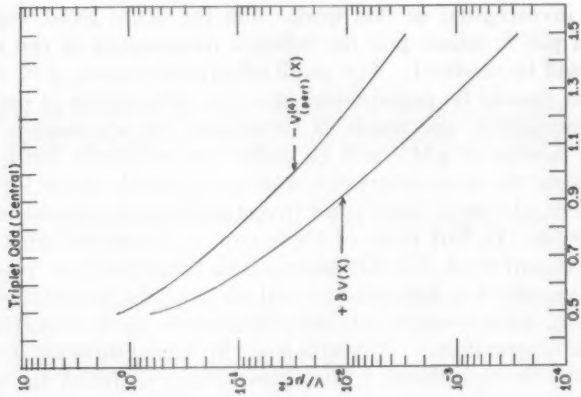

FIG. 3. Triplet odd central potential.

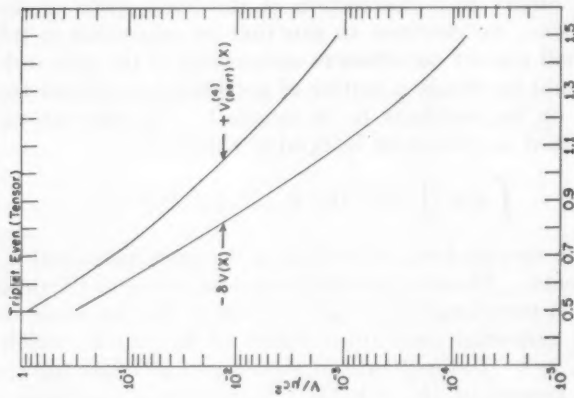

FIG. 2. Triplet even tensor potential.

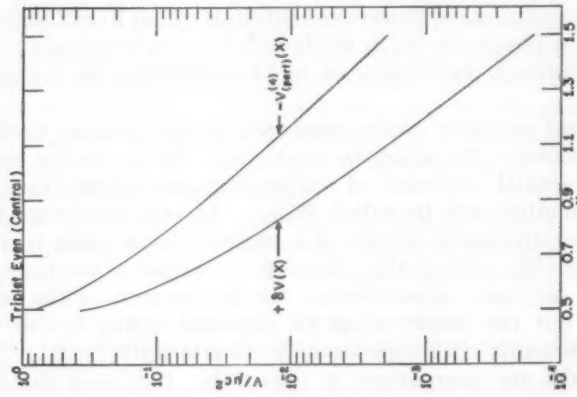

FIG. 1. Triplet even central potential.

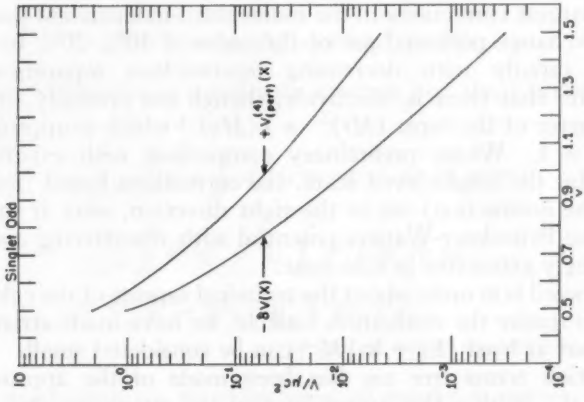

FIG. 6. Singlet odd potential.

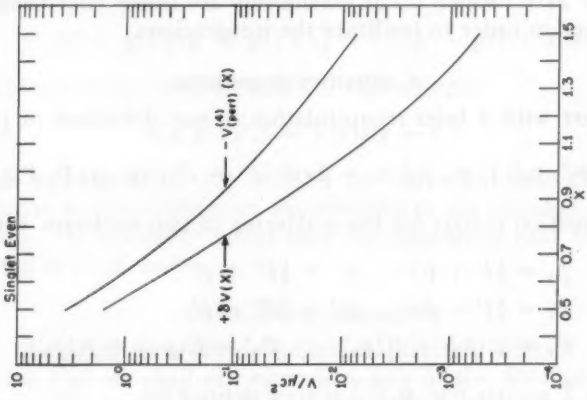

FIG. 5. Singlet even potential.

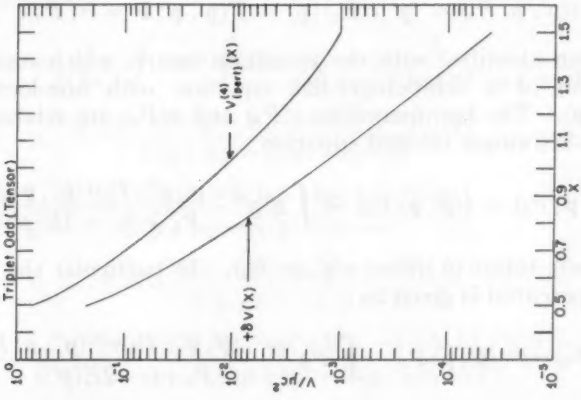

FIG. 4. Triplet odd tensor potential.

states the largest corrections to the momentum-independent part of the two-pion-exchange potential are of the order of 10%–20% for $r \sim r_0$, increasing rapidly with decreasing internucleon separation. This demonstrates that there is effectively, though not precisely, an expansion parameter of the form $(Mr)^{-1} \equiv \mu(Mx)^{-1}$ which is approximately 0.15 at $x = 1$. Where preliminary comparison with experiment is available, for the singlet even state, the corrections found (tending to diminish the interaction) are in the right direction, since it was found (8) that the Brueckner-Watson potential with rescattering corrections is too strongly attractive in this state.

A final word is in order about the technical aspects of the calculation. In order to render the evaluation feasible, we have made strong use of the idea that at least $|\mathbf{k}_1 + \mathbf{k}_2| M^{-1}$ can be considered small. Thus in the correction terms free use has been made of the approximation $\mathbf{k}_1 + \mathbf{k}_2 \approx 0$. Within this approximation the quantities $(\mathbf{k}_1^2 + M^2)^{1/2}$ and $(\mathbf{k}_2^2 + M^2)^{1/2}$ which occur extensively are freely (but consistently) interchanged in order to facilitate the integrations.

II. DEFINITION OF POTENTIAL

We start with a brief recapitulation of our definition of potential. Let

$$\langle p_1', p_2' | T | p_1, p_2 \rangle = -2\pi i \delta^4(P' - P) T(\mathbf{p}', \mathbf{p}; \mathbf{P}, P_0) \tag{2}$$

be the transition matrix for the scattering of two nucleons, with

$$\begin{aligned} p_1 &= \tfrac{1}{2}P + p; \qquad p_1' = \tfrac{1}{2}P' + p' \\ p_2 &= \tfrac{1}{2}P - p; \qquad p_2' = \tfrac{1}{2}P' - p' \\ P_0 &= E(p_1) + E(p_2); \qquad P_0' = E(p_1') + E(p_2') \end{aligned} \tag{3}$$

A reduced T matrix $t(\mathbf{p}', \mathbf{p}, P_0)$ is then defined by

$$t(\mathbf{p}', \mathbf{p}, P_0) = \langle \mathbf{p}' | t(P_0) | \mathbf{p} \rangle \equiv T(\mathbf{p}', \mathbf{p}; \mathbf{P} = 0; P_0). \tag{4}$$

$t(P_0)$ is then identified with the transition matrix which results from the solution of a Schrödinger-like equation with non-local kernel $\langle \mathbf{p}' | v(P_0) | \mathbf{p} \rangle$. The two quantities $t(P_0)$ and $v(P_0)$ are related by the Lippmann-Schwinger integral equation

$$v(\mathbf{p}', \mathbf{p}; P_0) = t(\mathbf{p}', \mathbf{p}; P_0) - \int d^3p'' \frac{v(\mathbf{p}', \mathbf{p}''; P_0) t(\mathbf{p}'', \mathbf{p}; P_0)}{P_0 + i\epsilon - 2E(p'')}, \tag{5}$$

which is here taken to define $v(\mathbf{p}', \mathbf{p}; P_0)$. In particular the two-pion exchange potential is given by

$$v^{(4)}(\mathbf{p}', \mathbf{p}; P_0) = t^{(4)}(\mathbf{p}', \mathbf{p}; P_0) - \int d^3p'' \frac{v^{(2)}(\mathbf{p}', \mathbf{p}''; P_0) v^{(2)}(\mathbf{p}'', \mathbf{p}; P_0)}{P_0 + i\epsilon - 2E(p'')}, \tag{6}$$

since

$$v^{(2)}(\mathbf{p}', \mathbf{p}; P_0) = t^{(2)}(\mathbf{p}', \mathbf{p}; P_0). \tag{7}$$

The potential kernel in coordinate space is just the Fourier transform of Eq. 6,

$$\begin{aligned} v(\mathbf{r}, \mathbf{r}'; P_0) &= \frac{1}{(2\pi)^3} \int d^3p \int d^3p' e^{i\mathbf{p}'\cdot\mathbf{r}'} v(\mathbf{p}', \mathbf{p}; P_0) e^{-i\mathbf{p}\cdot\mathbf{r}} \\ &= \frac{1}{(2\pi)^3} \int d^3(\mathbf{p}' - \mathbf{p}) \int d^3[\tfrac{1}{2}(\mathbf{p}' + \mathbf{p})] \\ &\quad \times V[(\mathbf{p}' - \mathbf{p}); \tfrac{1}{2}(\mathbf{p}' + \mathbf{p}); P_0] \\ &\quad \times e^{i(\mathbf{p}'-\mathbf{p})\cdot(1/2)(\mathbf{r}+\mathbf{r}')} e^{(1/2)(\mathbf{p}'+\mathbf{p})\cdot(\mathbf{r}'-\mathbf{r})}. \end{aligned} \tag{8}$$

The adiabatic limit, to which we shall restrict ourselves, is obtained by the approximation

$$V[(\mathbf{p}' - \mathbf{p}); \tfrac{1}{2}(\mathbf{p}' + \mathbf{p}); P_0] \approx V[(\mathbf{p}' - \mathbf{p}); 0; 2M] \tag{9}$$

which utilized in Eq. 8 shows that correspondingly

$$v(\mathbf{r}, \mathbf{r}'; P_0) \approx V(r)\delta(\mathbf{r} - \mathbf{r}'). \tag{10}$$

That the approximation (9) is valid for non-relativistic two-nucleon collisions can be seen only from examination of the detailed theoretical expressions. It is largely under such circumstances that the concept of potential is at all useful.

III. METHOD OF CALCULATION

Since our basic theoretical method is precisely that of Klein and McCormick (1a) we shall not reproduce any of the detailed theory of that paper. We begin the development with an equivalent of Eq. 44 from their paper.

$$\begin{aligned} t^{(4)}(\mathbf{p}', \mathbf{p}) = +\frac{i}{2(2\pi)^7} \int d^3p'' \int dk_0 \frac{1}{(k_0^2 - \omega_1^2 + i\epsilon)(k_0^2 - \omega_2^2 + i\epsilon)} \\ \times \Big\{ \frac{(\mathbf{p}' | j_\alpha^{(1)}(0) | \mathbf{p}' - \mathbf{k}_2)(\mathbf{p} + \mathbf{k}_1 | j_\beta^{(1)}(0) | \mathbf{p})}{E(\mathbf{p}+\mathbf{k}_1) - E(\mathbf{p}) + k_0 - i\epsilon} \\ + \frac{(\mathbf{p}' | j_\beta^{(1)}(0) | \mathbf{p}' - \mathbf{k}_1)(\mathbf{p} + \mathbf{k}_2 | j_\alpha^{(1)}(0) | \mathbf{p})}{E(\mathbf{p}+\mathbf{k}_2) - E(\mathbf{p}) - k_0 - i\epsilon} \Big\} \\ \times \Big\{ \frac{(-\mathbf{p}' | j_\alpha^{(2)}(0) | -\mathbf{p}' + \mathbf{k}_2)(-\mathbf{p} - \mathbf{k}_1 | j_\beta^{(2)}(0) | -\mathbf{p})}{E(\mathbf{p}+\mathbf{k}_1) - E(\mathbf{p}) - k_0 - i\epsilon} \\ + \frac{(-\mathbf{p}' | j_\beta^{(2)}(0) | -\mathbf{p}' + \mathbf{k}_1)(-\mathbf{p} - \mathbf{k}_2 | j_\alpha^{(2)}(0) | -\mathbf{p})}{E(\mathbf{p}+\mathbf{k}_2) - E(\mathbf{p}) + k_0 - i\epsilon} \Big\}. \end{aligned} \tag{11}$$

As mentioned in the introduction, this is not the complete expression for $t^{(4)}$, in that upon emission or absorption of a meson, each nucleon is restricted in its transitions to one nucleon intermediate states. This is the simplest approximation that can be utilized as a basis for the investigation of nonstatic effects.

According to Eq. 6 a subtraction of a form depending on $t^{(2)}$ must be made in order to define the potential. This subtraction is given by Eq. 45 of (1a). With the definitions,

$$\begin{aligned} E_1 &\equiv E(\mathbf{p}+\mathbf{k}_1) = E(\mathbf{p}'-\mathbf{k}_2) \\ E_2 &\equiv E(\mathbf{p}+\mathbf{k}_2) = E(\mathbf{p}'-\mathbf{k}_1) \\ E_0 &\equiv E(\mathbf{p}) \end{aligned} \tag{12}$$

and performing the integration over k_0, the following expression is then obtained for $v^{(4)}(\mathbf{p}', \mathbf{p})$:

$$\begin{aligned} v^{(4)}(\mathbf{p}', \mathbf{p}) = +\frac{1}{4(2\pi)^6}\int d^3p'' \\ \times \Bigg\{\!\!\Bigg\{ \frac{(\mathbf{p}'|j_\alpha^{(1)}(0)|\mathbf{p}'-\mathbf{k}_2)(\mathbf{p}+\mathbf{k}_1|j_\beta^{(1)}(0)|\mathbf{p})}{} \\ \times(-\mathbf{p}'|j_\alpha^{(2)}(0)|-\mathbf{p}'+\mathbf{k}_2)(-\mathbf{p}-\mathbf{k}_1|j_\beta^{(2)}(0)|-\mathbf{p}) \\ \Bigg\{\frac{1}{\omega_2(\omega_1^2-\omega_2^2)[\omega_2^2-(E_0-E_1)^2]} \\ -\frac{1}{\omega_1(\omega_1^2-\omega_2^2)[\omega_1^2-(E_0-E_1)^2]} \\ -\frac{\omega_1+\omega_2+E_0-E_1}{\omega_1\omega_2[\omega_1^2-(E_0-E_1)^2][\omega_2^2-(E_0-E_1)^2]}\Bigg\} \\ +(\mathbf{p}'|+j_\beta^{(1)}(0)|\mathbf{p}'-\mathbf{k}_1)(\mathbf{p}+\mathbf{k}_2|j_\alpha^{(1)}(0)|\mathbf{p}) \\ \times(-\mathbf{p}'|j_\beta^{(2)}(0)|-\mathbf{p}'+\mathbf{k}_1)(-\mathbf{p}-\mathbf{k}_2|j_\alpha^{(2)}(0)|-\mathbf{p}) \\ \times\Bigg\{\frac{1}{\omega_2(\omega_1^2-\omega_2^2)[\omega_2^2-(E_0-E_2)^2]} \\ -\frac{1}{\omega_1(\omega_1^2-\omega_2^2)[\omega_1^2-(E_0-E_2)^2]} \\ -\frac{\omega_1+\omega_2+E_0-E_2}{\omega_1\omega_2[\omega_1^2-(E_0-E_2)^2][\omega_2^2-(E_0-E_2)^2]}\Bigg\} \\ -[(\mathbf{p}'|j_\alpha^{(1)}(0)|\mathbf{p}'-\mathbf{k}_2)(\mathbf{p}+\mathbf{k}_1|j_\beta^{(1)}(0)|\mathbf{p}) \\ \times(-\mathbf{p}'|j_\beta^{(2)}(0)|-\mathbf{p}'+\mathbf{k}_1)(-\mathbf{p}-\mathbf{k}_2|j_\alpha^{(2)}(0)|-\mathbf{p}) \end{aligned}$$

(*Continued*)

$$+(\mathbf{p}'|j_\beta^{(1)}(0)|\mathbf{p}'-\mathbf{k}_1)(\mathbf{p}+\mathbf{k}_2|j_\alpha^{(1)}(0)|\mathbf{p})$$
$$\times(-\mathbf{p}'|j_\alpha^{(2)}(0)|-\mathbf{p}'+\mathbf{k}_2)(-\mathbf{p}-\mathbf{k}_1|j_\beta^{(2)}(0)|-\mathbf{p})]$$
$$\times\left\{\frac{1}{\omega_2(\omega_1^2-\omega_2^2)(\omega_2+E_1-E_0)(\omega_2+E_2-E_0)}\right.$$
$$\left.-\frac{1}{\omega_1(\omega_1^2-\omega_2^2)(\omega_1+E_1-E_0)(\omega_1+E_2-E_0)}\right\}\Big\}\Big\}. \quad (13)$$

Carrying out the transition to coordinate space and to the adiabatic limit, we are able to obtain after some change of variables

$$V^{(4)}(r)=+\frac{1}{2(2\pi)^6}\int d^3k_1\int d^3k_2 e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}$$
$$\times\Big\{\Big\{+(\mathbf{p}|j_\alpha^{(1)}(0)|\mathbf{p}-\mathbf{k}_1)(\mathbf{p}+\mathbf{k}_2|j_\beta^{(1)}(0)|\mathbf{p})$$
$$\times(-\mathbf{p}|j_\alpha^{(2)}(0)|-\mathbf{p}+\mathbf{k}_1)(-\mathbf{p}-\mathbf{k}_2|j_\beta^{(2)}(0)|-\mathbf{p})$$
$$\times\left\{\frac{1}{(\omega_1^2-\omega_2^2)}\left[\frac{1}{\omega_2[\omega_2^2-(E_0-E_1)^2]}-\frac{1}{\omega_1[\omega_1^2-(E_0-E_1)^2]}\right]\right.$$
$$\left.-\frac{\omega_1+\omega_2+E_0-E_1}{\omega_1\omega_2[\omega_1^2-(E_0-E_1)^2][\omega_2^2-(E_0-E_1)^2]}\right\}$$
$$-(\mathbf{p}|j_\alpha^{(1)}(0)|\mathbf{p}-\mathbf{k}_1)(\mathbf{p}+\mathbf{k}_2|j_\beta^{(1)}(0)|\mathbf{p})$$
$$\times(-\mathbf{p}|j_\beta^{(2)}(0)|-\mathbf{p}+\mathbf{k}_2)(-\mathbf{p}-\mathbf{k}_1|j_\alpha^{(2)}(0)|-\mathbf{p})$$
$$\times\left\{\frac{1}{(\omega_1^2-\omega_2^2)}\left[\frac{1}{\omega_2(\omega_2+E_2-E_0)(\omega_2+E_1-E_0)}\right.\right.$$
$$\left.\left.-\frac{1}{\omega_1(\omega_1+E_2-E_0)(\omega_1+E_1-E_0)}\right]\right\}\Big\}\Big\}. \quad (14)$$

Actually, the integrand of Eq. 14 still contains the full dependence on $\mathbf{p}$, but it is understood that after the evaluation of all matrix elements, the adiabatic limit is to be taken. It is not difficult to carry the calculation to terms linear in $\mathbf{p}$, but in accordance with the remarks of the introduction, we restrict ourselves henceforth to the zero order terms only. Equation 14 is the basis for all the remaining calculations of this paper.

Previous evaluations of Eq. 14, in the static limit $(\mu/M)\to 0$, have been based on the approximations

$$\mathbf{p}=0; \qquad E_0-E_1=0; \qquad E_0-E_2=0 \quad (15)$$

and typically

$$(\mathbf{p}|j_\alpha^{(1)}(0)|\mathbf{p}-\mathbf{k}_1)=i\left(\frac{f}{\mu}\right)\tau_\alpha^{(1)}\sigma^{(1)}\cdot\mathbf{k}_1. \quad (16)$$

Substitution of these values yields the expression

$$V_{st}^{(4)}(r) = -\frac{1}{2(2\pi)^6}\left(\frac{f}{\mu}\right)^4 \int d^3k_1 \int d^3k_2 e_i i^{(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}$$
$$\times \left\{ \frac{(3-2\tau^{(1)}\cdot\tau^{(2)})}{\omega_1^2\omega_2^2(\omega_1+\omega_2)} (\sigma^{(1)}\cdot\mathbf{k}_1)(\sigma^{(1)}\cdot\mathbf{k}_2)(\sigma^{(2)}\cdot\mathbf{k}_1)(\sigma^{(2)}\cdot\mathbf{k}_2) \right.$$
$$+ \frac{(3+2\tau^{(1)}\cdot\tau^{(2)})}{\omega_1\omega_2(\omega_1+\omega_2)} \left(\frac{1}{\omega_1^2}+\frac{1}{\omega_1\omega_2}+\frac{1}{\omega_2^2}\right)$$
$$\left. \times (\sigma^{(1)}\cdot\mathbf{k}_1)(\sigma^{(1)}\cdot\mathbf{k}_2)(\sigma^{(2)}\cdot\mathbf{k}_2)(\sigma^{(2)}\cdot\mathbf{k}_1) \right\}. \quad (17)$$

This is the result of Klein and McCormick (1a) and yields upon integration the Brueckner-Watson potential (2).

The principal inadequacy of this static limit lies in the fact that the approximations (15) and (16) assume that $k_1M^{-1} \ll 1$, $k_2M^{-1} \ll 1$. In what follows we shall only assume that $|\mathbf{k}_1+\mathbf{k}_2|\, M^{-1} \ll 1$. Indeed most correctly we should merely set $\mathbf{p}=0$ in Eq. 14 and then plow ahead to make an exact evaluation of what remains. This would entail extensive multiple numerical integrations. We have decided therefore, to countenance further approximation in our evaluation of the difference between Eqs. 14 and 17, the quantity $\delta V(r)$. According to Eq. 14 and the general properties of a Fourier integral, as long as we are interested in the value of $\delta V(r)$ for $r \gtrsim \mu^{-1}$, the most important contributions to Eq. 14 arise from $|\mathbf{k}_1+\mathbf{k}_2| \lesssim \mu$ and therefore $M^{-1}|\mathbf{k}_1+\mathbf{k}_2| \ll 1$, even with $\mathbf{k}_1$ and $\mathbf{k}_2$ individually large. We shall therefore wherever this simplifies the calculation, make the approximation $|\mathbf{k}_1|M^{-1} \approx |\mathbf{k}_2|M^{-1}$. In practice this will mean that the quantities $(\mathbf{k}_1^2+M^2)^{1/2}$ and $(\mathbf{k}_2^2+M^2)^{1/2}$, which are effectively the nucleon recoil energies, will be used interchangeably throughout the remainder of the calculation. This will mean in turn that the corrections calculated in this paper will themselves be most reliable for $r \gtrsim \mu^{-1}$ and least reliable for small internucleon separations.

IV. REDUCTION OF POTENTIAL TO FINAL FORM

In accordance with previous remarks the quantity to be investigated is

$$\delta V(r) = V^{(4)}(r) - V^{(4)}{}_{\text{static}}(r) \quad (18)$$

which represents the difference between the two-pion-exchange potential and its static limit. With the definitions

$$\Omega_{1,2} = (\mathbf{k}_{1,2}^2+M^2)^{1/2} \quad (19)$$

the salient approximations made in our work now will be detailed.

For the nucleon energies we write typically

$$E_0 = (\mathbf{p}^2 + M^2)^{1/2} = (\mathbf{p}'^2 + M^2)^{1/2} \approx M \tag{20}$$

$$E_1 = [(\mathbf{p} + \mathbf{k}_1)^2 + M^2]^{1/2} = [(\mathbf{p}' - \mathbf{k}_2)^2 + M^2]^{1/2} \approx \Omega_1 \approx \Omega_2 \approx \tfrac{1}{2}(\Omega_1 + \Omega_2) \tag{21}$$

reminding the reader for the last time that we shall work to zero order in $\mathbf{p}$. The determination of which form (21) to use will be dictated by convenience, the criterion of choice being that the resulting form permits one of the two integrals over $\mathbf{k}_1$ and $\mathbf{k}_2$ to be performed analytically.

A typical energy denominator in Eq. 14 may then be treated as follows

$$\begin{aligned}(\omega_1 + E_1 - E_0)^{-1} &\approx (\omega_1 - M + \Omega_1)^{-1} \\ &\equiv \frac{[1 + (\Omega_1/M) - \tfrac{1}{2}(\mu/M)^2]}{2\omega_1[1 - (\mu^2/2M\omega_1)]} - (2M)^{-1} \\ &\approx (2\omega_1)^{-1} + \Omega_1(2M\omega_1)^{-1} + \mu^2(4M\omega_1^2)^{-1} \\ &\quad + \mu^2\Omega_1(4M^2\omega_1^2)^{-1} - (2M)^{-1}. \end{aligned} \tag{22}$$

The expansion carried out in the last line of Eq. 22 is completely justified.

The remaining ingredient of the calculation that requires specification is the matrix element of $j_\alpha(0)$ between one-nucleon states. As is well known we have on invariance grounds

$$(\mathbf{p}\,|\,j_\alpha(0)\,|\,\mathbf{q}) = \left[\frac{M^2}{E(\mathbf{p})E(\mathbf{q})}\right]^{1/2} \mathbf{u}(p)(ig\gamma_5\tau_\alpha)u(q)K[(p-q)^2]. \tag{23}$$

Here $K[(p-q)^2]$ is a scalar function normalized to $K(-\mu^2) = 1$. g is then the renormalized pseudoscalar coupling constant. We shall replace K by unity since deviations from this value give rise to potentials of shorter range than those considered here according to the spectral representations for this quantity (**9**). Further evaluation then simply involves evaluation of Dirac Matrix elements of the type found in Eq. 23 and approximation in the same spirit as in Eq. 22.

We now record the final result of all these manipulations,

$$\delta V(r) = -\frac{1}{4(2\pi)^6}\left(\frac{f}{\mu}\right)^4 \lim_{r_1 \to r_2 \to r} \int d^3k_1 \int \frac{d^3k_2}{k_2^2}\, e^{i\mathbf{k}_1\cdot r_1} e^{i\mathbf{k}_2\cdot r_2} \times \Bigg\{ (3 - 2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}) \left[\frac{8M^2(\Omega_2 - M) - 4Mk_2^2}{\omega_1^3\omega_2^3}\right.$$

(*Continued*)

$$+\frac{4M(\Omega_2-M)-2k^2}{\omega_1^2\omega_2^2(\omega_1+\omega_2)}\Big]\times(\sigma^{(1)}\cdot\mathbf{k}_1)(\sigma^{(1)}\cdot\mathbf{k}_2)(\sigma^{(2)}\cdot\mathbf{k}_1)(\sigma^{(2)}\cdot\mathbf{k}_2)$$

$$+(3+2\tau^{(1)}\cdot\tau^{(2)})\left[\frac{4M(\Omega_2-M)-2k_2}{\omega_1\omega_2(\omega_1+\omega_2)}\right.$$

$$\times\left(\frac{1}{\omega_1^2}+\frac{1}{\omega_1\omega_2}+\frac{1}{\omega_2^2}\right)-\frac{4(\Omega_2-M)}{\omega_1^2\omega_2^2}$$

$$\left.+\frac{4\mu^2}{\omega_1^2\omega_2^2}\left(\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}\right)(\Omega_2-M)\right]$$

$$\times(\sigma^{(1)}\cdot\mathbf{k}_1)(\sigma^{(1)}\cdot\mathbf{k}_2)(\sigma^{(2)}\cdot\mathbf{k}_2)(\sigma^{(2)}\cdot\mathbf{k}_1)\Big\}. \quad (24)$$

It should be remarked that hereinafter it is not permissible to replace Ω_2 by Ω_1. Instead, inspection of the respective Taylor series shows that the following replacements, only, are permissible,

$$\frac{1}{k_2^2}(\Omega_2 - M) \leftrightarrow \frac{1}{k_1^2}(\Omega_1 - M)$$

$$\frac{1}{k_2^2}[4M(\Omega_2 - M) - 2k_2^2] \leftrightarrow \frac{1}{k_1^2}[4M(\Omega_1 - M) - 2k_1^2]. \quad (25)$$

V. NUMERICAL RESULTS AND CONCLUSIONS

Confining ourselves to the sanctioned approximations we have been able to evaluate (24) as a (huge) sum of terms involving either known functions (exponential or Hankel functions of imaginary argument) or single remaining numerical quadratures. The very lengthy details will not be given here. Detailed numerical study of some of the terms has convinced us that at this stage further judicious expansion in powers of M^{-1} is possible and keeping the leading terms only yields overall numerical accuracy of 5–10 per cent. Since we are already dealing with a correction term, we have considered this sufficient for our purposes. We shall therefore record only the final result after all such approximations have been made (10). We find

$$\delta V(x) \approx -\frac{f^4}{3m(2\pi)^4}(3 - 2\tau^{(1)}\cdot\tau^{(2)}$$

$$\times\left\{\left\{\right.\right. + \left(3 - \frac{9}{x^2}\right)[K_0(x)]^2 + \left(\frac{3}{x} - \frac{33}{x^3}\right)K_0(x)K_1(x)$$

$$\left.\left. + \left(\frac{6}{x^2} - \frac{30}{x^4}\right)[K_1(x)]^2\right\} - (\sigma^{(1)}\cdot\sigma^{(2)})\left\{ + \frac{2}{x^2}[K_0(x)]^2\right.\right.$$

(*Continued*)

$$+\left(\frac{10}{x^3}-\frac{4}{x}\right)K_0(x)K_1(x)+\frac{12}{x^4}[K_1(x)]^2\Big\}$$

$$+S_{12}\Big\{+\frac{1}{x^2}[K_0(x)]^2+\left(\frac{8}{x^3}-\frac{2}{x}\right)K_0(x)K_1(x)$$

$$+\left(\frac{12}{x^4}-\frac{3}{x^2}\right)[K_1(x)]^2\Big\}\Big\}$$

$$-\frac{f^4}{8(2\pi)^2}[m-(m^2-1)^{1/2}](3+2\tau^{(1)}\cdot\tau^{(2)})e^{-2x}$$

$$\times\Big\{\left(\frac{2}{x}+\frac{2}{x^2}-\frac{16}{x^4}-\frac{24}{x^5}-\frac{12}{x^6}\right)$$

$$+\tfrac{2}{3}(\sigma^{(1)}\cdot\sigma^{(2)})\left(-\frac{4}{x^2}+\frac{8}{x^4}+\frac{12}{x^5}+\frac{6}{x^6}\right)$$

$$+\tfrac{1}{3}S_{12}\left(\frac{4}{x^2}+\frac{6}{x^3}-\frac{8}{x^4}-\frac{24}{x^5}-\frac{12}{x^6}\right)\Big\}. \quad (26)$$

These results are compared with the Brueckner-Watson potential in Figs. 1–6.

From these results, it is possible to conclude that the perturbative potential is qualitatively correct, though it is quantitatively wrong. The first order corrections to this potential have proved, as expected, to be of relative order (μ/Mx) times the perturbative potential itself.

In conclusion, the present calculation uses the formulation of Klein and McCormick (**1a**) to obtain corrections to the perturbative two-pion-exchange potential. Of these corrections, the leading momentum independent parts have been evaluated, and have been shown to be non-negligible if quantitative agreement with experiment is sought. Before an attempt is made to refine further the contribution of one-nucleon intermediate states, at least two other types of contribution to the two-pion-exchange potential should be investigated. First, the contribution of intermediate states containing nucleon plus pion intermediate states should be investigated. Second, the contributions of pion-pion interactions require attention. Both of these phenomena could be treated best by an approach involving dispersion relations, suitably extrapolated off the mass shell. Finally, as the calculation of the two-pion-exchange potential is refined further, it will be necessary to obtain reliable results for the three-pion-exchange potential, which may be as large as some of the corrections to the two-pion-exchange potential.

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ON THE APPROXIMATE SOLUTION OF A CLASS OF STRONGLY ELLIPTIC LINEAR DIFFERENTIAL EQUATIONS *

BY

DONALD GREENSPAN [1]

SUMMARY

After a brief theoretical survey of the *analytical* knowledge available, a *finite-difference* procedure is described for the effective approximation of n-dimensional Dirichlet problems. The discussion is limited to linear elliptic partial differential equations and only the most elementary mathematical tools are used in establishing uniqueness and convergence for the method.

I. THEORETICAL PRELIMINARIES

Throughout, let G be a bounded domain in Euclidean n-space, that is, let G be an open, bounded, connected point set in E^n. Denote the boundary of G by $\dot{G}$ and the closure of G by $\bar{G}$. Let $f = f(x_1, x_2, \cdots, x_n)$; $a = a(x_1, \cdots, x_n)$; $a_i = a_i(x_1, x_2, \cdots, x_n)$, $i = 1, 2, \cdots, n$; $a_{ij} = a_{ij}(x_1, \cdots, x_n)$, $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, n$, be functions defined on G. Then the linear partial differential equation

$$\sum_i \sum_j a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i a_i \frac{\partial u}{\partial x_i} + au = f \tag{1}$$

is said to be elliptic on G if and only if, for all real $(\xi_1, \xi_2, \cdots, \xi_n)$, the quadratic form $\sum_i \sum_j a_{ij}\xi_i\xi_j$ is positive definite.

Definition 1. Equation 1 is said to be *strongly* elliptic on bounded domain G if and only if there exists a positive constant m such that, for all real $(\xi_1, \cdots, \xi_n)$, the positive definite quadratic form $\sum_i \sum_j a_{ij}\xi_i\xi_j$ satisfies the inequality

$$(\sum_i \sum_j a_{ij}\xi_i\xi_j) \geq (m \sum_i \xi_i^2). \tag{2}$$

In dealing with strongly elliptic linear differential equations, it will be desirable to restrict domain G and functional coefficients a_{ij}, a_i, a, f of Eq. 1 in the fashion indicated below.

Definition 2. A function F defined on a subset of E^n is said to satisfy a Hölder condition, or be Hölder continuous, if and only if there exist

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positive constants α and H, with $\alpha < 1$, such that for every pair of points P, Q of the indicated subset:

$$|F(P) - F(Q)| \leq H|P - Q|^{\alpha}. \tag{3}$$

In Definition 2, α is called the exponent and H is called the coefficient of the inequality.

Definition 3. For j a non-negative integer, bounded domain G is said to be of class L_j if and only if $\dot{G}$ can be covered by a finite number of open spheres in each of which $\dot{G}$ can be represented by an equation of the form

$$x_i = g(x_1, x_2, \cdots, x_{i-2}, x_{i-1}, x_{i+1}, x_{i+2}, \cdots, x_n);\ g \epsilon C^j.$$

Definition 4. For j a non-negative integer and α a fixed constant such that $0 < \alpha < 1$, bounded domain G is said to be of class $L_{j+\alpha}$ if and only if it is of class L_j and in each sphere the j^{th} partial derivatives of the corresponding function g satisfy a Hölder condition with exponent α.

For j a non-negative integer and α a real number which satisfies $0 < \alpha < 1$, the Banach spaces C_j and $C_{j+\alpha}$ with respect to bounded domain G are described as follows:

C_j: the space of functions which together with all their first j partial derivatives, are continuous on $\bar{G}$, the norm of any function F of C_j being defined by:

$$\|F\|_j = \max_{\bar{G}} |F| + \max_{\bar{G}} |DF| + \cdots + \max_{\bar{G}} |D^j F|,$$

where "$\max_{\bar{G}} |D^i F|$" is the maximum of the absolute values of all i^{th} order partial derivatives of F over all points of $\bar{G}$;

$C_{j+\alpha}$: the space of functions which, together with all the first j partial derivatives, are continuous on $\bar{G}$ and whose j^{th} partial derivatives are Hölder continuous on $\bar{G}$ with exponent α, the norm of any function $F \epsilon C_{j+\alpha}$ being defined by:

$$\|F\|_{j+\alpha} = \|F\|_j + \operatorname*{lub}_{P,Q \epsilon \bar{G}} \frac{|D^j F(P) - D^j F(Q)|}{|P - Q|^{\alpha}}.$$

The following theorem then is known ((1),[2] p. 60).

Theorem 1. On a bounded domain G of class $L_{2+\alpha}$, where $0 < \alpha < 1$, let Eq. 1 be strongly elliptic; let the functional coefficients f, a, a_i, a_{ij} be

[2] The boldface numbers in parentheses refer to the references appended to this paper.

in C_α; let $a \leq 0$; and let there exist a constant K such that

$$\|a_{ij}\|_\alpha, \|a_i\|_\alpha, \|a\|_\alpha \leq K.$$

Then if $\phi = \phi(x_1, x_2, \cdots, x_n)$ is continuous on $\dot{G}$, there exists a unique function $u = u(x_1, x_2, \cdots, x_n)$ which

(a) is a solution of Eq. 1 on G,
(b) is continuous on $\bar{G}$, and
(c) satisfies: $u(x_1, x_2, \cdots, x_n) + \phi(x_1, x_2, \cdots, x_n) \equiv 0$, on $\dot{G}$.

Theorem 1 gives sufficient conditions then for the *existence* of a unique solution of the n-dimensional Dirichlet problem for strongly elliptic linear partial differential equations. Within the contemporary mathematical framework, however, one usually does not know how to produce this solution. Assuming the conditions of Theorem 1, we proceed in the next section to describe a method for approximating the indicated unique solution for each of a *large class* of strongly elliptic equations (because of the complexities involved, equations with mixed partial derivatives will be treated elsewhere (2)). It should be noted also that Theorem 1 places severe restrictions on $\dot{G}$ which, for many important strongly elliptic equations, can be greatly relaxed (see (1), pp. 38, 44; (3), pp. 186–188).

II. METHOD OF APPROXIMATION

On bounded domain G of class $L_{2+\alpha}$, where $0 < \alpha < 1$, consider the partial differential equation

$$\sum_i a_{ii} \frac{\partial^2 u}{\partial x_i^2} + \sum_i a_i \frac{\partial u}{\partial x_i} + au = f. \tag{4}$$

Let Eq. 4 be strongly elliptic; let the functional coefficients $f = f(x_1, x_2, \cdots, x_n)$; $a = a(x_1, x_2, \cdots, x_n)$; $a_i = a_i(x_1, x_2, \cdots, x_n)$, $i = 1, 2, \cdots, n$; $a_{ii} = a_{ii}(x_1, x_2, \cdots, x_n)$, $i = 1, 2, \cdots, n$, be in C_α; let $a \leq 0$ on G; and let there exist a constant K such that

$$\|a_{ii}\|_\alpha, \|a_i\|_\alpha, \|a\|_\alpha \leq K. \tag{5}$$

Then if $\phi = \phi(x_1, x_2, \cdots, x_n)$ is continuous on $\dot{G}$, Theorem 1 implies that there exists a unique function $u = u(x_1, \cdots, x_n)$ which

(a) is a solution of Eq. 4 on G,
(b) is continuous on $\bar{G}$, and
(c) satisfies $u(x_1, \cdots, x_n) + \phi(x_1, \cdots, x_n) \equiv 0$ on $\dot{G}$.

A procedure for approximating $u(x_1, x_2, \cdots, x_n)$ will now be described in four steps. It is motivated by the fundamental ideas and

results of Liebmann (4), Gerschgorin (5), and Batschelet (6). Derivation of formulas, existence, uniqueness, and convergence questions will be treated in sequential sections.

Step 1. For $h_1, h_2, \cdots, h_n$ *positive* constants and for an arbitrary point $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$, determine the set of points of $\bar{G}$ of the form $(\bar{x}_1 + m_1h_1,\ \bar{x}_2 + m_2h_2, \cdots, \bar{x}_n + m_nh_n)$, where $m_1, m_2, \cdots, m_n$ may vary over all possible integral values. Denote this point set by $\bar{G}_h$.

Two points $(x_1, x_2, \cdots, x_n)$, $(y_1, y_2, \cdots, y_n)$ of $\bar{G}_h$ are called *adjacent* if and only if

(*a*) the straight line segment joining them is contained in $\bar{G}$, and

(*b*) $\sum_1^n (x_i - y_i)^2 = h_j^2$, where $x_i = y_i$ for all i *except* $i = j$.

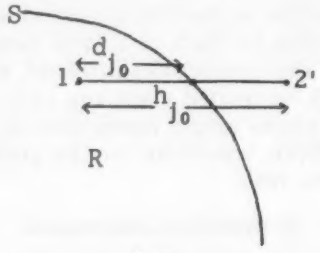

FIG. 1.

The interior of $\bar{G}_h$, denoted by G_h, is the set of all points of $\bar{G}_h$ which have $2n$ adjacent points in $\bar{G}_h$. The boundary of $\bar{G}_h$, denoted by $\dot{G}_h$, and called the lattice boundary, is defined by $\dot{G}_h = \bar{G}_h - G_h$.

Suppose then that $\bar{G}_h$ consists of t points. Number these in a one-to-one fashion with the positive integers 1, 2, 3, $\cdots$, t. At the point numbered k; $k = 1, 2, \cdots, t$, we shall denote the unknown function u by u_k and its approximation by U_k.

Step 2. Approximate u at each point of the lattice boundary as follows. Let $(x_1, x_2, \cdots, x_n)$ be an arbitrary, but fixed, point of $\dot{G}_h$ and let it be numbered 1. If $(x_1, x_2, \cdots, x_n)$ is also a point of $\dot{G}$, then set

$$U_1 = U(x_1, \cdots, x_n) = -\phi(x_1, x_2, \cdots, x_n). \tag{6}$$

If $(x_1, \cdots, x_n)$ is not a point of $\dot{G}$, then at least one of the $2n$ points $(x_1, x_2, \cdots, x_{j-1}, x_j \pm h_j, x_{j+1}, \cdots, x_n)$; $j = 1, 2, \cdots, n$, is not a point of $\bar{G}_h$ which is adjacent to $(x_1, x_2, \cdots, x_n)$. Suppose, without loss of generality, that one such point is $(x_1, x_2, \cdots, x_{j_0-1}, x_{j_0} + h_{j_0}, x_{j_0+1}, \cdots, x_n)$. Then there exists a point $(x_1, x_2, \cdots, x_{j_0-1}, x_{j_0} + d_{j_0}, x_{j_0+1}, \cdots, x_n)$,

$0 < d_{j_0} < h_{j_0}$, which we number $2'$ and which lies on $\dot{G}$, such that the straight line segment joining $(x_1, x_2, \cdots, x_n)$ and $(x_1, x_2, \cdots, x_{j_0-1}, x_{j_0} + d_{j_0}, x_{j_0+1}, \cdots, x_n)$ lies entirely in $\bar{G}$ (see Fig. 1). Then set

$$U_1 = U(x_1, x_2, \cdots, x_n) = -\phi(x_1, \cdots, x_{j_0-1}, x_{j_0}+d_{j_0}, x_{j_0+1}, \cdots, x_n). \quad (7)$$

Step 3. Approximate u at each point of G_h as follows. Let $(x_1, x_2, \cdots, x_n)$ be a point of G_h and let it be numbered 1. Then set

$$U_1 = \left[\frac{1}{a - 2\sum_i \frac{a_{ii}}{h_i^2}}\right]\left[f - \sum_i \left\{U_{1,h_i}\left[\frac{a_{ii}}{h_i^2} + \frac{a_i}{2h_i}\right] + U_{1,-h_i}\left[\frac{a_{ii}}{h_i^2} - \frac{a_i}{2h_i}\right]\right\}\right], \quad (8)$$

where

$$U_{1,h_i} = U(x_1, x_2, \cdots, x_{i-1}, x_i + h_i, x_{i+1}, \cdots, x_n) \quad (9)$$

$$U_{1,-h_i} = U(x_1, x_2, \cdots, x_{i-1}, x_i - h_i, x_{i+1}, \cdots, x_n). \quad (10)$$

and the functions f, a, a_i, a_{ii} are evaluated at $(x_1, x_2, \cdots, x_n)$.

Step 4. Application of Steps 1, 2, 3 yields a system of t linear algebraic equations in $U_1, U_2, \cdots, U_t$. The final step then is to solve this linear algebraic system for the respective approximations $U_1, U_2, \cdots, U_t$ of $u_1, u_2, \cdots, u_t$.

III. DERIVATION OF EQ. 8

Since Eq. 4 is strongly elliptic, there exists a positive constant m such that

$$\sum_i a_{ii}\xi_i^2 \geq m \sum_i \xi_i^2. \quad (11)$$

Setting $\xi_j = 1$; $\xi_k = 0$, $k \neq j$, implies with respect to (11), for the various cases $j = 1, 2, \cdots, n$, that

$$a_{ii} \geq m > 0; \qquad i = 1, 2, \cdots, n. \quad (12)$$

Note then that (12) and the assumption $a \leq 0$ imply

$$a - 2\sum_i \frac{a_{ii}}{h_i^2} < 0. \quad (13)$$

Now difference Eq. 8 is derived readily by substituting the central difference approximations (see (7), pp. 132, 140):

$$\left.\frac{\partial u}{\partial x_i}\right|_{(x_1, x_2, \cdots, x_n)} \sim \frac{U_{1,h_i} - U_{1,-h_i}}{2h_i}$$

$$\left.\frac{\partial^2 u}{\partial x_i^2}\right|_{(x_1, \cdots, x_n)} \sim \frac{U_{1,h_i} - 2U_1 + U_{1,-h_i}}{h_i^2}$$

into Eq. 4, by setting $u = U_1$, and by solving the result for U_1. The fact that one can actually solve for U_1 is a direct consequence of (13).

IV. UNIQUENESS OF THE APPROXIMATE SOLUTION

If the grid constants $h_1, h_2, \cdots, h_n$ are selected properly then, as indicated in the following theorem, the approximate solution described in Section II always exists and is unique. The techniques to be used in establishing this assertion, though not as sophisticated as those suggested in (8) and (9), will have the distinct advantage of relative simplicity.

Theorem 2. Since the $a_j(x_1, \cdots, x_n)$, $j = 1, 2, \cdots, n$, are continuous and hence bounded on $\bar{G}$, let c_j; $j = 1, 2, \cdots, n$ be non-negative constants such that on $\bar{G}$

$$|a_j(x_1, x_2, \cdots, x_n)| \leq c_j; \qquad j = 1, 2, \cdots, n. \tag{14}$$

Then if the positive grid constants $h_1, h_2, \cdots, h_n$ are selected so that

$$0 < c_i h_i < 2m; \qquad i = 1, 2, \cdots, n, \tag{15}$$

where m is given by (12), it follows that the solution of the linear algebraic system generated by the method of Section II exists and is unique.

Proof. Consider the linear system which results by selecting

$$f(x_1, x_2, \cdots, x_n) \equiv \phi(x_1, x_2, \cdots, x_n) \equiv 0. \tag{16}$$

This system is homogeneous and has the zero vector for a solution. The theorem will then be proved if it can be shown that the zero vector is the *only* solution, and this will now be done.

Since $\phi \equiv 0$ on $\dot{G}$, it follows from Eqs. 6 and 7 that if $(x_1, x_2, \cdots, x_n) \epsilon \dot{G}_h$, then $U(x_1, x_2, \cdots, x_n) = 0$. Now suppose there exists a non-trivial solution U for the homogeneous system. Then at some point of $\bar{G}_h$, $U \neq 0$. Suppose, without loss of generality, $U > 0$. Let the maximum value of U be M and let it occur at the point $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$, numbered 1 for the discussion, so that

$$M = U_1 = U(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) > 0, \tag{17}$$

$$M = U_1 = U(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \geq U(x_1, x_2, \cdots, x_n); \quad (x_1, x_2, \cdots, x_n) \epsilon \bar{G}_h. \tag{18}$$

Since $U = 0$ at each point of $\dot{G}_h$, it follows that $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \epsilon G_h$.

Relative to the points adjacent to $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$, the following two possibilities then arise:

(*i*) *Suppose all the* $U(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{i-1}, \bar{x}_j \pm h_j, \bar{x}_{i+1}, \cdots, \bar{x}_n)$; $j = 1, 2, \cdots, n$ *are equal.* Then, since $f \equiv 0$, Eq. 8 implies

$$U_1 = \frac{U_{1,h_1}}{\left(2\Sigma \frac{a_{ii}}{h_i^2}\right) - a}\left[2\Sigma \frac{a_{ii}}{h_i^2}\right] \leq U_{1,h_1}. \tag{19}$$

Then (17, 18, 19) imply

$$U_1 = U_{1,h_1} = M > 0. \tag{20}$$

(*ii*) *Suppose at least two of* U_{1,h_j}, $U_{1,-h_j}$, $j = 1, 2, \cdots, n$ *are not equal.* Then there exists a maximum. Without loss of generality, assume that U_{1,h_i} is that maximum. Hence

$$U_{1,h_j} = U_{1,h_1} - \alpha_j; \qquad j = 2, 3, \cdots, n \tag{21}$$

$$U_{1,-h_j} = U_{1,h_1} - \beta_j; \qquad j = 1, 2, \cdots, n, \tag{22}$$

where $\alpha_2, \alpha_3, \cdots, \alpha_n$; $\beta_1, \beta_2, \cdots, \beta_n$ are non-negative and at least one is positive. Substitution of (21) and (22) into Eq. 8, defining $\alpha_1 = 0$, and setting $\bar{a}_{ii} = a_{ii}(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$; $\bar{a}_i = a_i(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$; $\bar{a} = a(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ yield

$$U_1 = \left[\frac{1}{\left(2\Sigma \frac{\bar{a}_{ii}}{h_i^2}\right) - \bar{a}}\right]\left[\Sigma\left\{[U_{1,h_1} - \alpha_j]\left[\frac{\bar{a}_{ii}}{h_i^2} + \frac{\bar{a}_i}{2h_i}\right] + [U_{1,h_1} - \beta_j]\left[\frac{\bar{a}_{ii}}{h_i^2} - \frac{\bar{a}_i}{2h_i}\right]\right\}\right]. \tag{23}$$

Note that by (12) and (15)

$$\frac{\bar{a}_{ii}}{h_i^2} \pm \frac{\bar{a}_i}{2h_i} = \frac{1}{h_i^2}\left[\bar{a}_{ii} \pm \frac{\bar{a}_i h_i}{2}\right] > 0, \tag{24}$$

so that (13), (23), and (24) imply

$$U_1 = \frac{U_{1,h_1}}{\left(2\Sigma \frac{\bar{a}_{ii}}{h_i^2}\right) - \bar{a}}\left(\Sigma \frac{2\bar{a}_{ii}}{h_i^2}\right) - \frac{1}{\left(2\Sigma \frac{\bar{a}_{ii}}{h_i^2}\right) - \bar{a}}\left\{\Sigma\alpha_j\left[\frac{\bar{a}_{ii}}{h_i^2} + \frac{\bar{a}_i}{2h_i}\right] + \Sigma\beta_j\left[\frac{\bar{a}_{ii}}{h_i^2} - \frac{\bar{a}_i}{2h_i}\right]\right\}$$

(*Continued*)

$$= \frac{U_{1,k_1}}{\left(2\Sigma \frac{a_{ii}}{h_i^2}\right) - \bar{a}} \left(\Sigma \frac{2\bar{a}_{ii}}{h_i^2}\right) - S, \qquad (S > 0). \tag{25}$$

Hence

$$U_{1,k_1} = \frac{\left(2\Sigma \frac{\bar{a}_{ii}}{h_i^2}\right) - \bar{a}}{2\Sigma \frac{\bar{a}_{ii}}{h_i^2}} [U_1 + S] \geq M + S > M = U_1. \tag{26}$$

Thus, $U_1 < U_{1,k_1}$, which contradicts (18), and case (*ii*) is not possible.

Since case (*i*) is the only possibility, it follows that the argument may be repeated a finite number of times to show that

$$U(\tilde{x}_1 + mh_1, \tilde{x}_2, \cdots, \tilde{x}_n) = M,$$

where $(\tilde{x}_1 + mh_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ is a point of $\dot{G}_h$. But, by construction, $U(\tilde{x}_1 + mh_1, \tilde{x}_2, \cdots, \tilde{x}_n) = 0$ and a contradiction is reached.

Hence, the existence of the non-trivial solution $U(x_1, x_2, \cdots, x_n)$ is not possible and the theorem is proved.

Note that if the $a_i; i = 1, 2, \cdots, n$ are identically zero, as is the case in the Laplace equation, then one may select $c_i = 0, i = 1, 2, \cdots, n$ in (14). The $h_i; i = 1, 2, \cdots, n$ of (15) may then be selected arbitrarily without disturbing the validity of Theorem 2.

V. CONVERGENCE

The question of whether or not the approximate solution U converges to the analytical solution u will be explored by establishing an error bound and by showing, under certain conditions, that this error bound can be made arbitrarily small. Though, in accordance with (10), a limited amount of refinement *may* be available, the important problem of establishing a computable error bound remains, in general, unsolved.

First, the linear operators l_1 and l_2 are defined as follows:

$$l_1[v(x_1, x_2, \cdots, x_n)] = -v(x_1, x_2, \cdots, x_n) \tag{27}$$

$$\begin{aligned} l_2[v(x_1, \cdots, x_n)] = &\left[a - 2\Sigma \frac{a_{ii}}{h_i^2}\right] v(x_1, \cdots, x_n) \\ &+ \Sigma \left\{ v(x_1, \cdots, x_{i-1}, x_i + h_i, x_{i+1}, \cdots, x_n) \left[\frac{a_{ii}}{h_i^2} + \frac{a_i}{2h_i}\right] \right. \\ &\left. + v(x_1, \cdots, x_{i-1}, x_i - h_i, x_{i+1}, \cdots, x_n) \left[\frac{a_{ii}}{h_i^2} - \frac{a_i}{2h_i}\right] \right\}. \end{aligned} \tag{28}$$

Note that operators (27) and (28) enable one to set up bi-unique correspondences with subsets of $\bar{G}_h$ as follows:

$$l_1 \leftrightarrow \dot{G}_h \tag{29}$$

$$l_2 \leftrightarrow G_h. \tag{30}$$

In future reference, if operator l_i is considered at point $(x_1, x_2, \cdots, x_n)$, then it will be assumed that $(x_1, x_2, \cdots, x_n)$ belongs to the set defined by correspondences (29) and (30).

Lemma 1. For all $(x_1, x_2, \cdots, x_n)\epsilon\bar{G}_h$, let $z(x_1, x_2, \cdots, x_n)$ satisfy $l_i[z(x_1, x_2, \cdots, x_n)] \geq 0$; $i = 1, 2$. Then $z(x_1, x_2, \cdots, x_n) \leq 0$ on $\bar{G}_h$.

Proof. Suppose there exists a point of $\bar{G}_h$ at which $z > 0$. Then there exists a point $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$, at which z attains its maximum value, say M, on $\bar{G}_h$. Then

$$z(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) = M > 0 \tag{31}$$

$$z(x_1, x_2, \cdots, x_n) \leq M; \qquad (x_1, x_2, \cdots, x_n)\epsilon\bar{G}_h. \tag{32}$$

Two cases arise:

(*i*) If $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)\epsilon\dot{G}_h$, then

$$0 \leq l_1[z(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)] = -z(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) = -M, \tag{33}$$

so that $M \leq 0$. Thus, a contradiction of (31) implies $(x_1, x_2, \cdots, x_n)\epsilon G_h$.

(*ii*) If $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)\epsilon G_h$, then

$$\begin{aligned} 0 \leq l_2[z(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)] &= \left[\tilde{a} - 2\sum \frac{\tilde{a}_{ii}}{h_i^2}\right] z(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \\ &+ \sum \left\{ z(\bar{x}_1, \cdots, \bar{x}_{i-1}, \bar{x}_i + h_i, \bar{x}_{i+1}, \cdots, \bar{x}_n) \left[\frac{\tilde{a}_{ii}}{h_i^2} + \frac{\tilde{a}_i}{2h_i}\right] \right. \\ &\left. + z(\bar{x}_1, \cdots, \bar{x}_{i-1}, \bar{x}_i - h_i, \bar{x}_{i+1}, \cdots, \bar{x}_n) \left[\frac{\tilde{a}_{ii}}{h_i^2} - \frac{\tilde{a}_i}{2h_i}\right] \right\}. \end{aligned} \tag{34}$$

By the techniques of parts (*i*), (*ii*) of Theorem 2, inequality (34) implies that at some point of $\dot{G}_h$, $z = M$. This, however, reduces case (*ii*) to case (*i*), and the lemma follows by contradiction.

Lemma 2. If $v(x_1, x_2, \cdots, x_n)$, $w(x_1, x_2, \cdots, x_n)$ are defined on $\bar{G}_h$ and if $l_i[w] \leq l_i[v]$; $i = 1, 2$, at each point of $\bar{G}_h$, then $w(x_1, x_2, \cdots, x_n) \geq v(x_1, x_2, \cdots, x_n)$ on $\bar{G}_h$.

Proof. Let $z(x_1, x_2, \cdots, x_n) = v(x_1, x_2, \cdots, x_n) - w(x_1, x_2, \cdots, x_n)$. Recalling that l_1, l_2 are linear and applying Lemma 1 readily yields the proof.

Lemma 3. Let $v(x_1, x_2, \cdots, x_n)$, $w(x_1, x_2, \cdots, x_n)$ be defined on $\bar{G}_h$. If $l_i[w(x_1, x_2, \cdots, x_n)] \leq -|l_i[v(x_1, x_2, \cdots, x_n)]|$; $i = 1, 2$, at each point of $\bar{G}_h$, then $w(x_1, x_2, \cdots, x_n) \geq |v(x_1, x_2, \cdots, x_n)|$ on $\bar{G}_h$.

Proof. If $l_i[w] \leq -|l_i[v]|$, then $l_i[w] \leq l_i[v]$. By Lemma 2, then, $w \geq v$ on $\bar{G}_h$. However, $l_i[w] \leq -|l_i[v]|$ also implies $l_i[w] \leq -l_i[v]$, or, $l_i[w] \leq l_i[-v]$. Hence, $w \geq -v$ on $\bar{G}_h$. But $w \geq v$, $w \geq -v$ imply $w \geq |v|$, and the lemma is proved.

Further discussion will utilize the existence of a suitable amount of differentiability on the analytical solution $u(x_1, x_2, \cdots, x_n)$. Hence, it will be assumed in the remainder of the discussion that $u \epsilon C^4$ on G and that all partial derivatives of u up to and including order four are uniformly bounded on G. A convenient notation, in this connection, will be

$$M_\alpha = \text{l.u.b.} \max_\beta \left| \frac{\partial^\alpha u}{\partial x_i^\beta \partial x_j^{\alpha-\beta}} \right|, \qquad (x_1, x_2, \cdots, x_n) \epsilon G. \tag{35}$$

We define now the error function E by

$$E(x_1, x_2, \cdots, x_n) = u(x_1, x_2, \cdots, x_n) - U(x_1, x_2, \cdots, x_n). \tag{36}$$

With the aid of finite Taylor expansions, it follows from (7) and (27) that if $(x_1, x_2, \cdots, x_n)$ is a point of $\dot{G}_h$ but is *not* a point of $\dot{G}$, then

$$\begin{aligned}
l_1[E] &= l_1[u(x_1, \cdots, x_n) - U(x_1, \cdots, x_n)] \\
&= l_1[u] - l_1[U] \\
&= -u(x_1, \cdots, x_n) + \phi(x_1, x_2, \cdots, x_{j_0-1}, x_{j_0} + d_{j_0}, x_{j_0+1}, \cdots, x_n) \\
&= \phi(x_1, x_2, \cdots, x_{j_0-1}, x_{j_0} + d_{j_0}, x_{j_0+1}, \cdots, x_n) - u(x_1, \cdots, x_n) \\
&= \phi(x_1, x_2, \cdots, x_{j_0-1}, x_{j_0} + d_{j_0}, x_{j_0+1}, \cdots, x_n) \\
&\qquad - \Big[u(x_1, \cdots, x_{j_0-1}, x_{j_0} + d_{j_0}, x_{j_0+1}, \cdots, x_n) \\
&\qquad\qquad - d_{j_0} \frac{\partial u}{\partial x_{j_0}} \Big| (x_1, \cdots, x_{j_0-1}, \xi, x_{j_0+1}, \cdots, x_n) \Big] \\
&= d_{j_0} \frac{\partial u}{\partial x_{j_0}} \Big| (x_1, \cdots, x_{j_0-1}, \xi, x_{j_0+1}, \cdots, x_n)
\end{aligned}$$

Hence

$$|l_1[E]| \leq h_{j_0} M_1. \tag{37}$$

If $(x_1, \cdots, x_n) \epsilon (\dot{G}_h \cap \dot{G})$, then $l_1[E] = 0$, so that (37) is valid for all points of $\dot{G}_h$.

With the aid of infinite Taylor expansions, it follows from (1), (8) and (28) that

$$
\begin{aligned}
l_2[E] &= l_2[u] - l_2[U] \\
&= \left[a - 2\Sigma \frac{a_{ii}}{h_i^2}\right] u(x_1, \cdots, x_n) \\
&\quad + \Sigma \left\{ u(x_1, \cdots, x_{i-1}, x_i + h_i, x_{i+1}, \cdots, x_n) \left[\frac{a_{ii}}{h_i^2} + \frac{a_i}{2h_i}\right] \right. \\
&\quad \left. + u(x_1, \cdots, x_{i-1}, x_i - h_i, x_{i+1}, \cdots, x_n) \left[\frac{a_{ii}}{h_i^2} - \frac{a_i}{2h_i}\right] \right\} \\
&\quad - f(x_1, \cdots, x_n) \\
&= \left[a - 2\Sigma \frac{a_{ii}}{h_i^2}\right] u(x_1, \cdots, x_n) \\
&\quad + \Sigma \left\{ \left[u(x_1, \cdots, x_n) + h_i \frac{\partial u}{\partial x_i}\bigg|_{(x_1, \cdots, x_n)} \right.\right. \\
&\quad + \frac{h_i^2}{2!} \frac{\partial^2 u}{\partial x_i^2}\bigg|_{(x_1, \cdots, x_n)} + \frac{h_i^3}{3!} \frac{\partial^3 u}{\partial x_i^3}\bigg|_{(x_1, \cdots, x_n)} \\
&\quad \left.\left. + \frac{h_i^4}{4!} \frac{\partial^4 u}{\partial x_i^4}\bigg|_{(x_1, \cdots, \gamma_i^{(1)}, \cdots, x_n)} \right] \left[\frac{a_{ii}}{h_i^2} + \frac{a_i}{2h_i}\right] \right. \\
&\quad + \left[u(x_1, \cdots, x_n) - h_i \frac{\partial u}{\partial x_i}\bigg|_{(x_1, \cdots, x_n)} \right. \\
&\quad + \frac{h_i^2}{2!} \frac{\partial^2 u}{\partial x_i^2}\bigg|_{(x_1, \cdots, x_n)} - \frac{h_i^3}{3!} \frac{\partial^3 u}{\partial x_i^3}\bigg|_{(x_1, \cdots, x_n)} \\
&\quad \left.\left. \left. + \frac{h_i^4}{4!} \frac{\partial^4 u}{\partial x_i^4}\bigg|_{(x_1, \cdots, \gamma_i^{(2)}, \cdots, x_n)} \right] \left[\frac{a_{ii}}{h_i^2} - \frac{a_i}{2h_i}\right] \right\} - f(x_1, \cdots, x_n) \\
&= \left[\Sigma a_{ii} \frac{\partial^2 u}{\partial x_i^2} + \Sigma a_i \frac{\partial u}{\partial x_i} + au - f\right] \\
&\quad + \Sigma \left[\frac{h_i^2}{3!} a_i \frac{\partial^3 u}{\partial x_i^3}\bigg|_{(x_1, \cdots, x_n)} \right. \\
&\quad + \frac{h_i^2}{4!} a_{ii} \frac{\partial^4 u}{\partial x_i^4}\bigg|_{(x_1, \cdots, \gamma_i^{(1)}, \cdots, x_n)} \\
&\quad \left. + \frac{h_i^3}{2 \cdot 4!} a_i \frac{\partial^4 u}{\partial x_i^4}\bigg|_{(x_1, \cdots, \gamma_i^{(1)}, \cdots, x_n)} \right.
\end{aligned}
$$

(*Continued*)

$$+\frac{h_i^2}{4!}a_{ii}\frac{\partial^4 u}{\partial x_i^4}\Big|(x_1,\cdots,\gamma_i^{(3)},\cdots,x_n)$$
$$-\frac{h_i^3}{2\cdot 4!}a_i\frac{\partial^4 u}{\partial x_i^4}\Big|(x_1,\cdots,\gamma_i^{(3)},\cdots,x_n)\Big]$$
$$=\Sigma\Big[\frac{h_i^2}{3!}a_i\frac{\partial^3 u}{\partial x_i^3}\Big|(x_1,\cdots,x_n)$$
$$+\frac{h_i^2}{4!}a_{ii}\frac{\partial^4 u}{\partial x_i^4}\Big|(x_1,\cdots,\gamma_i^{(1)},\cdots,x_n)$$
$$+\frac{h_i^3}{2\cdot 4!}a_i\frac{\partial^4 u}{\partial x_i^4}\Big|(x_1,\cdots,\gamma_i^{(1)},\cdots,x_n)$$
$$+\frac{h_i^2}{4!}a_{ii}\frac{\partial^4 u}{\partial x_i^4}\Big|(x_1,\cdots,\gamma_i^{(2)},\cdots,x_n)$$
$$-\frac{h_i^3}{2\cdot 4!}a_i\frac{\partial^4 u}{\partial x_i^4}\Big|(x_1,\cdots,\gamma_i^{(2)},\cdots,x_n)\Big]. \tag{38}$$

Now, since the a_{ii} are bounded on $\bar{G}$, let

$$|a_{ii}| \leq c_{ii}; \qquad i = 1, 2, \cdots, n \tag{39}$$

where the c_{ii} are positive real constants. Then (14), (35), (38), and (39) imply

$$|l_2[E]| \leq \frac{M_3}{6}\Sigma h_i^2 c_i + \frac{M_4}{12}\Sigma c_{ii}h_i^2 + \frac{M_4}{24}\Sigma c_i h_i^3. \tag{40}$$

Theorem 3. If the analytical solution $u = u(x_1, x_2, \cdots, x_n)$ of the problem stated in Section 1 is of class C^4 on G and if the partial derivatives of u up to and including order four are uniformly bounded on G, then the approximate solution U, described in Section 2, converges to the analytical solution u as $(h_1^2 + h_2^2 + \cdots + h_n^2)$ converges to zero.

Proof. Let $\bar{u}(x_1, \cdots, x_n)$ be the analytical solution of the problem described in Section 1 with $\phi \equiv f \equiv -1$. Hence, on G

$$\Sigma a_{ii}\frac{\partial^2 \bar{u}}{\partial x_i^2} + \Sigma a_i\frac{\partial \bar{u}}{\partial x_i} + a\bar{u} \equiv -1, \tag{41}$$

while on $\dot{G}$

$$\bar{u}(x_1, \cdots, x_n) - 1 \equiv 0. \tag{42}$$

We will now relate $E(x_1, \cdots, x_n)$ to $\bar{u}(x_1, \cdots, x_n)$. This will be done by selecting m, independently of $x_1, \cdots, x_n$ so that on $\bar{G}_h$

$$|E(x_1, x_2, \cdots, x_n)| \leq w(x_1, \cdots, x_n) = m\bar{u}(x_1, \cdots, x_n); \qquad m \geq 0. \quad (43)$$

Now, (43) would be valid by Lemma 3 on $\bar{G}_h$ if

$$l_i[w] \leq -|l_i[E]|; \qquad i = 1, 2. \quad (44)$$

Consider, therefore, (44) for $i = 1, 2$, in an effort to determine a suitable $m \geq 0$ for (43).

Case 1. *Suppose* $(x_1, \cdots, x_n) \in \dot{G}_h$.

If $(x_1, \cdots, x_n)$ is not an element of $\dot{G}$, then

$$\begin{aligned} l_1[w(x_1, \cdots, x_n)] &= -w(x_1, \cdots, x_n) \\ &= -\left[w(x_1, \cdots, x_{j_0-1}, x_{j_0} + d_{j_0}, x_{j_0+1}, \cdots, x_n) \right. \\ &\qquad \left. - d_{j_0} \frac{\partial w}{\partial x_{j_0}}\bigg|_{(x_1, \cdots, \xi, \cdots, x_n)}\right] \\ &= -\left[m\bar{u}(x_1, \cdots, x_{j_0-1}, x_{j_0} + d_{j_0}, \cdots, x_n) \right. \\ &\qquad \left. - d_{j_0} m \frac{\partial \bar{u}}{\partial x_{j_0}}\bigg|_{(x_1, \cdots, \xi, \cdots, x_n)}\right] \\ &= -\left[m \cdot 1 - d_{j_0} m \frac{\partial \bar{u}}{\partial x_{j_0}}\right] \\ &= -m\left[1 - d_{j_0} \frac{\partial \bar{u}}{\partial x_{j_0}}\right]. \end{aligned}$$

If we restrict ourselves to h_{j_0} such that

$$d_{j_0} \text{ l.u.b.} \left|\frac{\partial \bar{u}}{\partial x_{j_0}}\right| \leq h_{j_0} \text{ l.u.b.} \left|\frac{\partial \bar{u}}{\partial x_{j_0}}\right| < \frac{1}{2}, \quad (45)$$

then

$$l_1[w] \leq -\frac{m}{2}. \quad (46)$$

However, from (37),

$$|l_1[E]| \leq h_{j_0} M_1,$$

so that if one selects

$$m \geq 2h_{j_0} M_1, \quad (47)$$

then (44) is valid.

If, on the other hand, $(x_1, \cdots, x_n) \in (\dot{G}_h \cap \dot{G})$, then $l_1[w] = -m$, and the selection of m according to (47) again assures the validity of (44).

Case 2. Suppose $(x_1, \cdots, x_n) \in G_h$.

Then, as in the development of (38),

$$
\begin{aligned}
l_2[w(x_1, \cdots, x_n)] &= m\left[\sum a_{ii}\frac{\partial^2 \bar{u}}{\partial x_i^2} + \sum a_i \frac{\partial \bar{u}}{\partial x_i} + a\bar{u}\right] \\
&\quad + m\left[\sum \frac{h_i^2}{3!} a_i \frac{\partial^3 \bar{u}}{\partial x_i^3}\right|_{(x_1, \cdots, x_n)} \\
&\quad + \frac{h_i^2}{4!} a_{ii} \frac{\partial^4 \bar{u}}{\partial x_i^4}\Big|_{(x_1, \cdots, \delta_i^{(1)}, \cdots, x_n)} \\
&\quad + \frac{h_i^3}{2\cdot 4!} a_i \frac{\partial^4 \bar{u}}{\partial x_i^4}\Big|_{(x_1, \cdots, \delta_i^{(1)}, \cdots, x_n)} \\
&\quad + \frac{h_i^2}{4!} a_{ii} \frac{\partial^4 \bar{u}}{\partial x_i^4}\Big|_{(x_1, \cdots, \delta_i^{(2)}, \cdots, x_n)} \\
&\quad - \frac{h_i^3}{2\cdot 4!} a_i \frac{\partial^4 \bar{u}}{\partial x_i^4}\Big|_{(x_1, \cdots, \delta_i^{(2)}, \cdots, x_n)}\Bigg] \\
&= -m + m\left[\sum \frac{h_i^2}{3!} a_i \frac{\partial^3 \bar{u}}{\partial x_i^3}\right|_{(x_1, \cdots, x_n)} \\
&\quad + \frac{h_i^2}{4!} a_{ii} \frac{\partial^4 \bar{u}}{\partial x_i^4}\Big|_{(x_1, \cdots, \delta_i^{(1)}, \cdots, x_n)} \\
&\quad + \frac{h_i^3}{2\cdot 4!} a_i \frac{\partial^4 \bar{u}}{\partial x_i^4}\Big|_{(x_1, \cdots, \delta_i^{(1)}, \cdots, x_n)} \\
&\quad + \frac{h_i^2}{4!} a_{ii} \frac{\partial^4 \bar{u}}{\partial x_i^4}\Big|_{(x_1, \cdots, \delta_i^{(2)}, \cdots, x_n)} \\
&\quad - \frac{h_i^3}{2\cdot 4!} a_i \frac{\partial^4 \bar{u}}{\partial x_i^4}\Big|_{(x_1, \cdots, \delta_i^{(2)}, \cdots, x_n)}\Bigg].
\end{aligned}
$$

If we restrict ourselves to $h_1, h_2, \cdots h_n$ such that

$$
\begin{aligned}
&\frac{1}{6}\left[\text{l.u.b.}\left|\frac{\partial^3 \bar{u}}{\partial x_i^3}\right|\right][\sum h_i^2 c_i] + \frac{1}{12}\left[\text{l.u.b.}\left|\frac{\partial^4 \bar{u}}{\partial x_i^4}\right|\right][\sum c_{ii} h_i^2] \\
&\qquad + \frac{1}{24}\left[\text{l.u.b.}\left|\frac{\partial^4 \bar{u}}{\partial x_i^4}\right|\right][\sum c_i h_i^3] < \frac{1}{2}, \qquad (48)
\end{aligned}
$$

then

$$
l_2[w] \leq -\frac{m}{2}. \qquad (49)
$$

However, from (40),

$$|l_2[E]| \leq \frac{M_3}{6} \sum h_i^2 c_i + \frac{M_4}{12} \sum c_{ii} h_i^2 + \frac{M_4}{12} \sum c_i h_i^3,$$

so that if one selects

$$m \geq \frac{M_3}{6} \sum h_i^2 c_i + \frac{M_4}{12} \sum c_{ii} h_i^2 + \frac{M_4}{24} \sum c_i h_i^3,$$

then (44) is valid for $i = 2$.

Thus, if one confines himself only to values of $h_1, h_2, \cdots h_n$ which satisfy both (45) and (48), then choosing m by

$$m = \max\left[2h_{j_0} M_i; \frac{M_3}{6} \sum h_i^2 c_i + \frac{M_4}{12} \sum c_{ii} h_i^2 + \frac{M_4}{24} \sum c_i h_i^3\right] \quad (50)$$

implies the validity of (44) and hence of (43), so that

$$|E(x_1, \cdots, x_n)| \leq \bar{u}(x_1, \cdots, x_n) \left\{\max\left[2h_{j_0} M_i; \frac{M_3}{6} \sum h_i^2 c_i + \frac{M_4}{12} \sum c_{ii} h_i^2 + \frac{M_4}{24} \sum c_i h_i^3\right]\right\}. \quad (51)$$

Finally, since $\bar{u}(x_1, \cdots, x_n)$ is continuous and hence bounded on $\bar{G}$, it follows from (51) that if $(h_1^2 + \cdots + h_n^2) \to 0$, then $|E| \to 0$, that is, $U \to u$, and the theorem is proved.

VI. CONCLUDING REMARK

It is worth noting, in conclusion, that the first order method of approximation indicated in Step 2, Section 2, Eq. 5 may be replaced by the following second order, linear interpolation scheme (11). Let $(x_1, x_2, \cdots, x_n)$ be an arbitrary, but fixed, point of $\dot{G}_h$ and let it be numbered 1. If $(x_1, x_2, \cdots, x_n)$ is not a point of $\dot{G}$, then at least one of the $2n$ points $(x_1, x_2, \cdots, x_{j-1}, x_j \pm h_j, x_{j+1}, \cdots, x_n)$; $j = 1, 2, \cdots, n$ is not a point of $\dot{G}_h$ which is adjacent to $(x_1, \cdots, x_n)$. Suppose, without loss of generality, that one such point is $(x_1, x_2, \cdots, x_{j_0-1}, x_{j_0} + h_{j_0}, x_{j_0+1}, \cdots, x_n)$. Then there exists a point $(x_1, x_2, \cdots, x_{j_0-1}, x_{j_0} + d_{j_0}, x_{j_0+1}, \cdots x_n)$, $0 < d_{j_0} < h_{j_0}$, which we number $2'$ and which lies on $\dot{G}$, such that the straight line segment joining $(x_1, x_2, \cdots, x_n)$ and $(x_1, x_2, \cdots, x_{j_0-1}, x_{j_0} + d_{j_0}, x_{j_0+1}, \cdots, x_n)$ lies entirely in $\bar{G}$. Then two cases arise according as the point $(x_1, x_2, \cdots, x_{j_0-1}, x_{j_0} - h_{j_0}, x_{j_0+1}, \cdots, x_n)$ is, or is not, a point of $\bar{G}_h$ which is adjacent to $(x_1, x_2, \cdots, x_n)$.

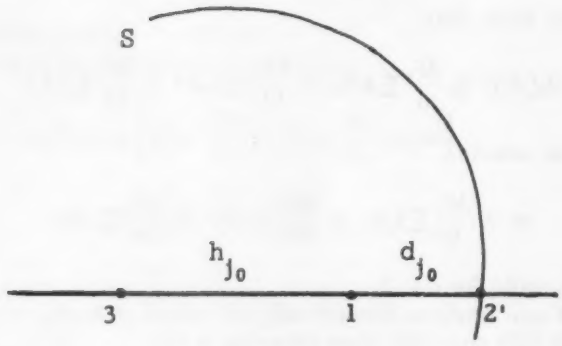

FIG. 2.

Case 1. Suppose $(x_1, x_2, \cdots, x_{j_0-1}, x_{j_0} - h_{j_0}, x_{j_0+1}, \cdots, x_n)$ is a point of $\bar{G}_h$ which is adjacent to $(x_1, x_2, \cdots, x_n)$. Then, as in Fig. 2, let $(x_1, x_2, \cdots, x_{j_0-1}, x_{j_0} - h_{j_0}, x_{j_0+1}, \cdots, x_n)$ be numbered 3 and approximate u_1 by

$$U_1 = \frac{d_{j_0} U_3 + h_{j_0} U_2'}{d_{j_0} + h_{j_0}}. \tag{52}$$

By Eq. 7, however, $U_2' = -\phi(x_1, \cdots, x_{j_0-1}, x_{j_0} + d_{j_0}, x_{j_0+1}, \cdots, x_n)$, so that Eq. 52 may be rewritten as

$$U_1 = \frac{d_{j_0} U_3 - h_{j_0}\phi(x_1, \cdots, x_{j_0-1}, x_{j_0} + d_{j_0}, x_{j_0+1}, \cdots, x_n)}{h_{j_0} + d_{j_0}}. \tag{53}$$

Case 2. Suppose $(x_1, x_2, \cdots, x_{j_0} - h_{j_0}, x_{j_0+1}, \cdots, x_n)$ is not a point of $\bar{G}_h$ which is adjacent to $(x_1, x_2, \cdots, x_n)$. Then there exists a point $(x_1, x_2, \cdots, x_{j_0-1}, x_{j_0} - d_{j_0}', x_{j_0+1}, \cdots, x_n)$, $0 < d_{j_0}' < h_{j_0}$, which we

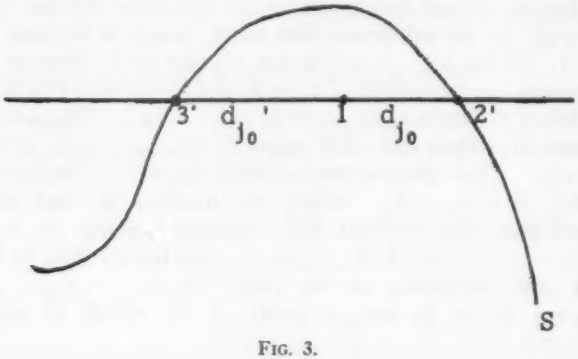

FIG. 3.

number $3'$ and which lies on $\dot{G}$, such that the straight line segment joining $(x_1, x_2, \cdots, x_n)$ and $(x_1, x_2, \cdots, x_{j_0-1}, x_{j_0} - d_{j_0}', x_{j_0+1}, \cdots, x_n)$ lies entirely in $\bar{G}$ (see Fig. 3). Then approximate u_1 by

$$U_1 = \frac{d_{j_0} U_3' + d_{j_0}' U_2'}{d_{j_0} + d_{j_0}'}$$

or, equivalently, by

$$U_1 = \frac{\begin{array}{r} - d_{j_0}\phi(x_1, x_2, \cdots, x_{j_0-1}, x_{j_0} - d_{j_0}', x_{j_0+1}, \cdots, x_n) \\ - d_{j_0}'\phi(x_1, x_2, \cdots, x_{j_0-1}, x_{j_0} + d_{j_0}, x_{j_0+1}, \cdots, x_n) \end{array}}{d_{j_0} + d_{j_0}'}. \quad (54)$$

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THE OPTIMUM DENSITY OF PRIOR EXPOSURE

BY

CHRIS CANDLER [1]

SUMMARY

When the primary exposure is infinitesimal, the supplementary exposure is an optimum, when the curvature d^2D/dE^2 of the characteristic is zero. Applying an equation of the characteristic to a typical plate, this equation yields a density of 0.8, a value four times the density used in practice. The theory is amended to cover a primary exposure comparable with the supplementary, and the predicted optimum shown to agree much better with practice.

I. CURRENT THEORY

Two authors (3, 7)[2] have shown that when the primary exposure is infinitesimal, the supplementary exposure is an optimum at the inflection of the *D-E* characteristic.

$$d^2\mathbf{D}/dE^2 = 0 \tag{1}$$

Here **D** is the natural density ln $(1/T)$, and T is the transmittance. True, Branscomb (3) gave the condition as

$$d^2\mathbf{D}/d(\ln E)^2 = d\mathbf{D}/d(\ln E) \tag{2}$$

but as

$$d^2\mathbf{D}/d(\ln E)^2 = d\mathbf{D}/d(\ln E) + E^2 d^2\mathbf{D}/dE^2 \tag{3}$$

is an identity, the point of inflection is implied.

These authors attacked the problem by very different roads. Branscomb (3) made the signal-to-noise ratio a maximum; while the writer accepted the Kaiser equations (12) for superposed exposures. So when the primary exposure is infinitesimal, the optimum density is well established. Unfortunately neither author reminded his readers that the problem he had solved is highly academic. For both assumed the primary exposure tends to zero, a condition never satisfied in practice; for if it were, the image would still be invisible after intensification. Indeed the primary exposure is often as large as the supplementary.

In generalizing the earlier theory, the density will be assumed a single valued function of the exposure, for only so can the argument be made rigid, while remaining reasonably simple. This assumption implies that the reciprocity law is valid and the sequence effect negligible. How far these assumptions are permissible is discussed in a final section.

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[2] The boldface numbers in parentheses refer to the references appended to this paper.

II. NOMENCLATURE

Nomenclature is still a difficulty. When the density is a single valued function of the exposure, the density due to two exposures is the same whether A follows B or B follows A, so the German "aufeinanderfolgenden Doppelbelichtungen" is logical; but even those who proposed it (11) would hardly claim it to be as brief as it is accurate. To achieve brevity, some American authors (20, 16) coined the omnibus word "latensification" by condensing "latent image intensification." But if the expanded form is allowed to linger in the mind, this introduces an unreal distinction between prior and post exposure, for prior exposure cannot properly be described as intensification at all. So latensification is best confined to chemical processes.

When two exposures are given, the whole plate receives a supplementary exposure E_s, while the object of interest, say a faint spectrum, records a "primary" exposure E_p. The "supplementary" exposure is then the "prior" exposure of common parlance.

The density produced by the supplementary exposure alone is D_s and that by the primary exposure alone D_p, while after the plate has received both exposures, the resultant density is D, and the "useful" density D_u of the spectrum above background

$$D_u = D - D_s. \tag{4}$$

As the total exposure E is sum of the parts,

$$E = E_s + E_p, \tag{5}$$

the points (E_s, D_s), (E_p, D_p) and (E, D) all lie on the normal characteristic of the plate, but the point (E_p, D_u) lies on quite a different curve obtained from the normal curve by displacing the origin from $(0, 0)$ to (E_s, D_s) (7).

III. AN EQUATION OF THE CHARACTERISTIC

The optimum density can be properly discussed only when an equation of the characteristic is available. At the present day three equations are in use. The first

$$A + P \log E = PKD + \log (1 - T) \tag{6}$$

derives from the Kaiser (15, 5) transform, P being the photon number and K the ratio of the absorption cross sections of a halide crystal before and after development. This equation has been widely used by spectrochemists (1, 2, 17, 18) and astronomers (22) as well as in a theoretical study (13). When fitted to an experimental curve, this equation can scarcely be distinguished from the other two, for the unavoidable error in fitting the parameters P, K and A often exceeds the difference

between the equations. The Kaiser equation, however, has no foundation in theory.

The second equation

$$A + P \log E = PKD + \log D \tag{7}$$

was based on theory of a kind (6), but it need not further detain us, because it is inferior to a third (8)

$$A + P \log E = \log (10^{PKD} - 1) \tag{8}$$

on both experimental and theoretical grounds. This third equation is properly ascribed to Hurter (14, 10).

At low densities the author's equation may be written

$$(sE)^P = \mathbf{B} + \mathbf{B}^2 + \mathbf{B}^3/2 + \cdots \tag{9}$$

and Hurter's

$$(sE)^P = \mathbf{B} + \tfrac{1}{2}\mathbf{B}^2 + \mathbf{B}^3/6 + \cdots \tag{10}$$

where s is the reaction cross section of a halide crystal, and $\mathbf{B}$ is an abbreviation for $PK\mathbf{D}$. Equations 9 and 10 are so similar that experiment cannot distinguish them at low densities, and in this paper only low densities concern us.

In theory Hurter's equation is superior because it derives from Hurter's differential equation

$$K\delta D = (e^{-K\mathbf{D}} - R^{-1})\delta(sE). \tag{11}$$

Here R is the "opacity before development," or the exposure ratio of front to back surface (6). Because Hurter thought the photon number unity—*not*, be it noted, because he overlooked the issue—he omitted it from his equation. When this omission is corrected, and Hurter's equation applied to an emulsion so deep that no light escapes from the back surface,

$$K\delta D = e^{-PK\mathbf{D}}(sE)^{P-1}\delta(sE). \tag{12}$$

On integration this yields the equation of the deep emulsion

$$(sE)^P = e^{PK\mathbf{D}} - 1. \tag{13}$$

Historically this argument is of interest as demonstrating Hurter's uncanny intuition, for he had little on which to rely.

Today the Hurter equation may be derived by quite a different argument. The exposure equation of a monolayer is established by the immaculate grain counts of Sheppard, Trivelli and Loveland (19)

in the form (8)

$$(sE)^P = n/(1 - n) \tag{14}$$

where n is the fraction of the grains fertilized by an exposure E. When the monolayer is integrated in depth, a procedure (9, 4, 21) no longer open to dispute, the Hurter equation of a deep emulsion emerges.

Of the dozen equations proposed for the photographic characteristic, Hurter's alone is able to explain monolayer, Schumann plate and coating by a self-consistent set of equations, so it stands in a class by itself.

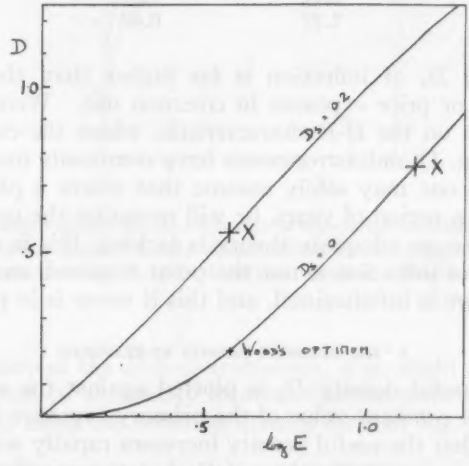

FIG. 1. The characteristic after prior exposure to a density of 0.2 shown derived from the normal characteristic by displacement of the origin. The exposure equation assumed is $P \log E = \log (10^{PKD} - 1)$ with $P = 2.4$ and $PK = 0.5$. X is the point of inflection on both curves.

IV. THE DENSITY AT INFLECTION

Given the Hurter equation of a deep emulsion, the slope

$$d\mathbf{B}/d(sE) = P(1 - e^{-\mathbf{B}})/sE \tag{15}$$

and the curvature

$$d^2\mathbf{B}/d(sE)^2 = P(1 - e^{-\mathbf{B}})(Pe^{-\mathbf{B}} - 1/(sE)^2 \tag{16}$$

of the $\mathbf{B}$-E curve follow by differentiation. At inflection

$$e^{\mathbf{B}} = P \tag{17}$$

so the common density D_x at this point is

$$D_x = (PK)^{-1} \log P. \tag{18}$$

Four Kodak plates were exposed to monochromatic radiation of wave length 436 $m\mu$ in 1956 and measured at Harrow. The Hurter equation has been fitted to these characteristics, and the parameters determined (see Table I).

TABLE I.

Plate	P	PK	D_z
B 10	3.28	0.50	1.03
B 20	2.30	0.53	0.68
L 20	2.44	0.40	0.97
L 15	2.22	0.60	0.58

The density D_z at inflection is far higher than the densities of supplementary or prior exposure in common use. Wood (23) recommended a point on the D-E characteristic, where the curve begins to rise steeply (Fig. 1) and astronomers have commonly used densities of 0.1 or 0.2. As one may safely assume that where a physicist uses a procedure over a period of years, he will recognize the optimum conditions, even when an adequate theory is lacking, this is clear evidence that the point of inflection is not the point required, except when the primary exposure is infinitesimal, and this it never is in practice.

V. THE OPTIMUM DENSITY RE-EXAMINED

When the useful density D_u is plotted against the supplementary density D_s for a constant value of the primary exposure E_p, the curves (Fig. 2) show that the useful density increases rapidly with the supplementary exposure for small values of D_s, but at a supplementary of 0.2 the curves flatten out, so the low densities used by astronomers are justified.

To complete the solution, note that each $D_u - D_s$ curve passes through a maximum determined by the condition

$$(\partial D/\partial E_s)_{E_p} = 0 \tag{19}$$

for the maximum will occur at the same point, whether the independent variable is D_s or E_s. On applying this condition to Eq. 5,

$$dD_s/dE_s = (\partial D/\partial E_s)_{E_p}. \tag{20}$$

As E is the sum of E_p and E_s, and here E_p is constant, this is equivalent to

$$dD_s/dE_s = dD/dE. \tag{21}$$

So the slope of the D-E characteristic after the supplementary exposure alone is equal to the slope after it has been superposed on the primary exposure, and the tangents at the two points are parallel (Fig. 3).

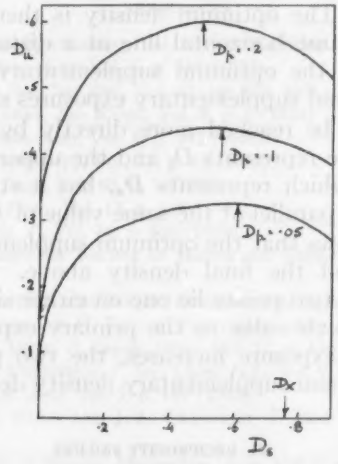

FIG. 2. The useful density D_u as a function of the background density D_s for various values of E_p. D_p is the primary density without supplementary exposure. An arrow shows the maximum on each curve, and D_x is the density at inflection. The curve assumes the same equation as Fig. 1.

The curvature of the *D-E* characteristic is so slight that the points of contact cannot be accurately located, but the slope dD/dE may be plotted (Fig. 4) using the common density form of Eq. 17

$$dD/dE = (1 - 10^{-PKD})/MKE, \tag{22}$$

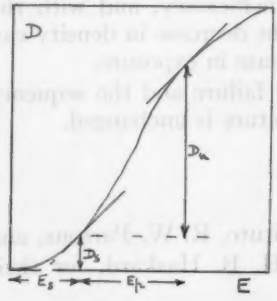

FIG. 3. The supplementary exposure is an optimum when the two tangents are parallel. The curvature of the *D–E* curve is here much exaggerated.

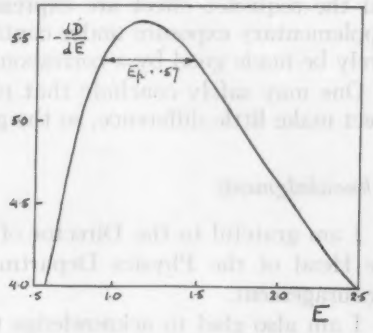

FIG. 4. The slope of the *D–E* curve used in Fig. 1.

where $M = 0.434$. The optimum density is then found by locating two points on the same horizontal line at a distance E_p apart. The lower value of E is the optimum supplementary exposure, and the higher the primary and supplementary exposures superposed.

This result may be reached more directly by considering Fig. 1, for if the lower curve represents D_s and the upper D, the distance between the curves, which represents D_u, has a stationary value only when the curves are parallel at the same value of E.

This solution shows that the optimum supplementary exposure lies below inflection, and the final density above. When the primary density is small, the two points lie one on either side of inflection, and tend to the Branscomb value as the primary exposure tends to zero. When the primary exposure increases, the two points move farther apart, and the optimum supplementary density decreases.

VI. RECIPROCITY FAILURE

To simplify the argument in the preceding sections, the density has been assumed a single valued function of the exposure, so that reciprocity failure and the sequence effect are excluded.

The same assumption is implicit, whenever the total exposure is taken to be the sum of the parts (Eq. 5), and thus also in the spectrochemists' method of correcting for background (12). Originally the Kaiser rule was stated rather than proved, but it has been so widely used that it is now accepted as correct.

True, in spectrochemistry the sequence effect is non-existent, since both exposures are contemporaneous; but in the present context this assumption is scarcely more objectionable, for both reciprocity failure and the sequence effect are expressions of inefficiency, and with the supplementary exposure under control, a slight decrease in density can surely be made good by a corresponding increase in exposure.

One may safely conclude that reciprocity failure and the sequence effect make little difference, so the general picture is unchanged.

Acknowledgments

I am grateful to the Director of the Institute, R. W. Parsons, and the Head of the Physics Department, G. H. B. Haskard, for their encouragement.

I am also glad to acknowledge the four monochromatic characteristics recorded in the Kodak Research Laboratories at Harrow.

This is also the place to express my regret that in an earlier paper I overlooked Branscomb's work.

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READER OPINION

CORRECTIONS TO "GRAVITATIONAL TORQUE ON A SATELLITE VEHICLE" *

BY

ROBERT E. ROBERSON [1]

It has been pointed out to the author by Joseph Dunn of Grumman Aircraft Corporation that Eq. 8 of the subject paper gives incorrect results for large attitude motions of the satellite. He suggests correctly that this is because I fell into the same trap from which I have often warned students, namely, using the gradient operator improperly in skewed co-ordinates.

The results should be corrected as follows. It is desired to find the components of $\bar{L}$ in the principal axis system XYZ. It is known that $-\frac{\partial U}{\partial \theta_i} = \bar{L}\cdot\bar{e}_{\theta i}$ where $\bar{e}_{\theta i}$ is the unit vector along the axis of rotation of the angle θ_i. Denoting $\bar{L}$ by the component representation $L_1\bar{X} + L_2\bar{Y} + L_3\bar{Z}$ and forming the scalar product successively with $\bar{x}$, $(\bar{X}\sin\theta_3 + \bar{Y}\cos\theta_3)$, $\bar{Z}$, one obtains the three scalar equations

$$\left.\begin{aligned} L_1\cos\theta_3\cos\theta_2 - L_2\cos\theta_2\sin\theta_3 &= \frac{\partial U}{\partial \theta_1} \\ L_1\sin\theta_3 + L_2\cos\theta_3 &= -\frac{\partial U}{\partial \theta_2} \\ L_3 &= -\frac{\partial U}{\partial \theta_3} \end{aligned}\right\} \tag{1}$$

from which

$$\left.\begin{aligned} L_1 &= -\frac{\partial U}{\partial \theta_1}\sec\theta_2\cos\theta_3 - \frac{\partial U}{\partial \theta_2}\sin\theta_3 + \frac{\partial U}{\partial \theta_3}\tan\theta_2\cos\theta_3 \\ L_2 &= \frac{\partial U}{\partial \theta_1}\sec\theta_2\sin\theta_3 - \frac{\partial U}{\partial \theta_2}\cos\theta_3 - \frac{\partial U}{\partial \theta_3}\tan\theta_2\sin\theta_3 \\ L_3 &= -\frac{\partial U}{\partial \theta_3} \end{aligned}\right\} \tag{2}$$

These are to be compared with Eqs. 9 of the reference, from which they differ only by second-order terms in small quantities if the θ_i are small angles.

Since only the case of small angles is treated explicitly in the reference in Eqs. 10 ff, it follows that this work is still valid. However, Eqs. 9 must be modified as above if the work is to be applied to the large angle case.

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* JOUR. FRANKLIN INST., Vol. 265, pp. 13–22 (1958).

[1] Consultant in Astronautics, Fullerton, Calif.

BOOK REVIEWS

LECTURES ON DIFFERENTIAL AND INTEGRAL EQUATIONS, by Kosaku Yosida. 220 pages, 6 × 9 in. New York, Interscience Publishers, Inc., 1960. Price, $7.00.

In about two hundred pages, Professor Yosida (University of Tokyo) gives a self-contained exposition of the theory of differential and integral equations. Numerous old and new treatises deal with the content of this book with variable emphases and approaches. This little book strikes one by its coherence and lucidity.

In the first chapter, the author exposes the method of successive approximations, the general theory of linear differential equations, and that of the second order equations of the Fuchs type, in particular Gauss, Legendre and Bessel equations. The main properties of the special functions are indicated while the reader is sent to specialized works for more elaborate identities. In the second chapter the Boundary value problem for Linear Differential equations of the second order is considered. An elegant treatment is given for the Green's function and the reduction to integral equations. This is followed by the Hilbert-Schmidt theory of integral equations with symmetric kernels. Liowville's treatment is used to derive an asymptotic expression for the Eigen value and Eigen functions. In Chapters 3 and 4, an expose' is given of Fredholm and Volterra types of integral equations. Chapter 5 is devoted to the more recent expansion theorem of Weyl-Stone-Titschmanoh-Kodaira. Despite its long name, this theorem is very elegant and the modern density matrix method proves to be extremely useful in numerous branches of Mathematical Physics.

The book is written in a compact and clear style, rigorous without being unduly so, and will be enjoyed by mathematicians and mathematical physicists alike. The lack of proposed problems is probably the only regrettable feature of this book.

HENRI AMAR
Temple University

ANALOGUE COMPUTATION, by Stanley Fifer. Four volumes, 1331 pages, illustrations, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $39.50.

The author of this four-volume work has succeeded in providing a comprehensive coverage of all facets of the electronic analog computer art. Although he is President of Dian Laboratories, Inc., a manufacturer of analog computing equipment, Mr. Fifer has presented an unbiased book which uses as illustrations the products of all of the prominent analog computer manufacturers. It appears that the emphasis of the book is primarily on computer "hardware," although the author does present much of the theoretical background which is associated with the effective utilization of these devices; particularly in the final volume. While all of the material is very clearly written and well illustrated, this writer cannot help feeling that the several of the topics could have been better treated in a more concise manner.

Volumes one and two are devoted to the general-purpose d-c. amplifier analog computer and its associated input-output equipment which constitutes the type of machine present in an overwhelming majority of analog installations. Each of the individual components—the d-c. feedback amplifier, the potentiometer, the multiplier, the resolver, the diode, the relay, the function generator, etc.—is covered in a separate chapter. Also presented in volume two are chapters dealing with the topics of checking and error analysis, transfer functions, and various selected techniques or "tricks-of-the-trade" which enable the user to extend the versatility of the computer, as well as chapters which illustrate the programming of various types of nonlinear differential equations.

Volumes three and four discuss some of the broader ramifications of "analogue computation," and deal with such topics as network analyzers, mechanical differential analyzers, special computers such as magnetic, repetitive, and thermal devices, partial differential

equations, fourier analysis, noise, linear simultaneous algebraic equations, and various advanced techniques such as sampled-data control systems, applications to automation and process control, and others.

In summary, this four-volume set is recommended to the individual whose work and interests are associated with analog computers. This book makes a valuable addition to the library of such a person as a handbook which may suggest a more effective way to solve a number of problems, since it fully presents a discussion of analogue techniques in their broadest sense.

W. H. STEIGELMANN
The Franklin Institute Laboratories

AN INTRODUCTION TO TRANSITION-METAL CHEMISTRY, by Leslie E. Orgel. 180 pages, diagrams, 5¼ × 8¼ in. New York, John Wiley & Sons, Inc., 1960. Price, $4.50.

In this monograph the author, an active investigator in this field, attempts to give an account of ligand-field theory which is "both short and non-mathematical." After a few definitions and preliminaries, the author gives a qualitative discussion of the atomic *s*, *p*, and *d* orbitals, their energies and their symmetries. He then discusses ions with several *d*-electrons and their stereochemistry. This is followed by a good and clear discussion of the stability of transition-metal compounds, ionic radii, energy levels and spectra. Reaction mechanisms and rates are briefly considered, but the following chapter on the covalent and ionic bonds and on low valency transition metals are excellent. The book is concluded by a discussion of unsaturated hydrocarbons and high valency transition elements.

This monograph provides certainly interesting and instructive reading on a phase of chemistry that is rapidly growing. One wonders, however, for whom it has been written. The author views it as giving "a general impression" to students of inorganic chemistry. To this reader, it seems that students that are unfamiliar with Pauling's "Nature of the Chemical Bond" or equivalent treatments, may be frustrated by this qualitative and non-mathematical account. As for those who have this appropriate background, (in particular Pauling's treatment of the metallic bond), they will want a more thorough exposition of this interesting and specialized field.

HENRI AMAR
Temple University

SPACEFLIGHT TECHNOLOGY, edited by Kenneth W. Gatland. 365 pages, illustrations, 6 × 9 in. London, Academic Press, 1960. Price, $11.00.

In August 1959 just prior to the meeting of the International Astronautical Federation in London, the British Interplanetary Society staged a highly successful three-day symposium in which British Commonwealth scientists were invited to present papers.

There was a tremendous response to the invitations and the symposium, if judged by the papers and the caliber of the men who presented them, was assured of success.

There is something a little sad about the symposium in that the sponsors are men with vision and foresight who, had funds been available, would have been in the forefront of astronautic research. Unfortunately, the vast sums of money required for this type of an effort is unavailable to these scientists and the only avenue left them was theoretical researches. Thus the papers which were presented were related to theory and to feasibility studies of mating various forms of rockets in the British rocket arsenal as launching vehicles. Instead of presenting papers detailing researches performed by rocket and satellite studies the scientists were limited to studies.

There is one other unfortunate aspect of this situation. As matters now stand there is destined to be a space program involving cooperative efforts by several European powers. Because of the costs they will use launch vehicles supplied by the United States. The use of these launch vehicles will vitiate one important phase of this work for had the British and other countries been able to develop their own launch vehicles the addition of their inputs would have materially enhanced the over-all space picture. By giving these scientists hardware already in existence we do not permit them to exercise their judgments, resources and imagination to the solution of the problems which arise.

Spaceflight Technology represents the pro-

ceedings of the symposium. The content ranges from the mundane involving programs and economics to launching vehicles, the re-entry problem, propulsion systems, space vehicle cabins, instrumentation, tracking and communications, navigation and the exploration of the moon.

There are several significant papers in this book. The one which is most striking is the Hilton paper given by H. R. Watson involving recovery using aerodynamic lift. While this work was performed in 1957 its importance is striking today in view of the work being pursued on Dyna-Soar. Winged re-entry into the atmosphere is a necessary condition for the fulfillment of the Dyna-Soar objective. Here some three or more years ago the ground work was being laid by a British scientist. This analysis indicates that a three-stage vehicle with an all-up weight of 160 tons (an Atlas on the launching pad weighs over 125 tons) would put a two-man pyramid vehicle, weighing about two tons into orbit. The concept is close to that being used today in this country to put the Mercury astronaut into space.

Cabin conditioning equipment by George Beardshall and Peter W. Fitt explores the problem of a sealed capsule atmosphere and is based in a large measure on researches in the United States. The state of the art has passed the point where this is representative of current knowledge. In a measure this is true of other papers in this book. In scientific instrumentation of unmanned earth satellites by Barratt, Rothwell and Somes-Charlton we are given a comprehensive survey of the tools used in space exploration by unmanned satellites but here again the paper while of extreme interest to the researcher falls short of being definitive because of the time element and the lack of work in operating rocket probes.

Of tremendous interest to this reviewer is the Carton paper on soft moon-landing of instruments. With the Surveyor probes but two years away, this paper is timely and covers fields being researched today in the United States. The author indicates that with an all-up wieght of from 250,000–350,000 lb. they look for a 240-lb. payload on the moon. In the Surveyor project they will use the Atlas-Centaur launch vehicle of the all-up weight indicated in the study but the Surveyor will put 750 lb. on the moon of which 200 lb. will be instruments.

Spaceflight Technology is a guide to the astronautical thinking of the British Commonwealth up to the summer of 1959.

I. M. LEVITT
The Fels Planetarium

ATOMS TO GALAXIES, by James Stokley. 351 pages, illustrations, and diagrams, 5¾ × 9 in. New York, Ronald Press, 1961. Price, $6.00.

It can be said that two types of people write popular books on science. One is the professional scientist who, as a general rule, is magnificently inarticulate in trying to tell his story to the lay public. The others are the pseudo scientists who, as a general rule, do not completely understand the science so that they give a popularized version which is devoid of true meaning. Thus our popular scientific books fall short of the goal established by either the scientist or the writer. Occasionally, there comes along a scientist with the capability of translating science to the layman. These men are jewels and to them we owe a debt of gratitude. These middlemen between science and the public are desperately needed in a scientifically oriented world.

The author, James Stokley, is one of these rare individuals who has spent many years in science and journalism. His ability to make science understandable is recognized by both camps. Beginning his career as the first Director of the Fels Planetarium, he pioneered in science popularization. By combining this with a skillful and facile pen, he is among the leaders in the writing of popular science books. His latest work is *Atoms to Galaxies*.

With this comprehensive title he can "free wheel" through the Universe and this is precisely what he does—most effectively. As the title indicates, the author literally goes from atoms to galaxies.

The first four chapters deal with background so that what follows can be understood. The atom, radiations, radioastronomy and nuclear reactions set the stage for a discussion of the solar system. The author then devotes chapters to the sun, earth and the moon. Mercury, Venus and Mars are covered in a chapter and the planets beyond Mars are also

collected in a chapter. At the conclusion of this last chapter is a section on life on other planets. An excellent chapter is devoted to comets and meteors.

Chapter 11 is a magnificant bit of exposition. This section on travel in space is superbly done. While the entire field cannot be covered, the author wisely selected those topics of extreme interest and in keeping with the format of the book.

The next three chapters involve the stars, interstellar material and the galaxies of the stars. The final chapter deals with the question of the expanding universe versus the steady state one.

Atoms and Galaxies is one of the most readable introductions to modern astronomy this reviewer has seen. It is the ideal book for the layman who would know more about the fascinating universe.

I. M. LEVITT
The Fels Planetarium

PUBLICATIONS RECEIVED

THE ATOM AND ITS NUCLEUS, by George Gamow. 153 pages, diagrams, 5½ × 8 in. Englewood Cliffs (N. J.), Prentice-Hall, Inc., 1961. Price: $3.75 (cloth); $1.95 (paper).

THE NATURE OF THERMODYNAMICS, by P. W. Bridgman. Harper Torchbook 537. Enlarged and reprinted from the 1941 edition, 239 pages, 5¼ × 8 in. New York, Harper & Brothers, 1961. Price, $1.40 (paper).

CAUSALITY AND CHANCE IN MODERN PHYSICS, by Davie Bohm, with foreword by Louis De Broglie. Harper Torchbook 536. Reprinted from the 1957 edition, 170 pages, 5¼ × 8 in. New York, Harper & Brothers, 1961. Price, $1.35 (paper).

TURNING POINTS IN PHYSICS, by R. J. Blin-Stoyle, D. ter Haar, K. Mendelssohn, G. Temple, F. Waismann, and D. H. Wilkinson. Harper Torchbook 535. Reprinted from the 1959 edition, 192 pages, 5¼ × 8 in. New York, Harper & Brothers, 1961. Price, $1.45 (paper).

CHEMICAL PROPERTIES AND IDENTIFICATION OF IONS. INTRODUCTION TO SEMIMICROQUALITATIVE ANALYSIS, by Omer K. Whipple. 310 pages, diagrams, 6 × 8 in. San Francisco, Wadsworth Publishing Co., 1961. No price.

KINETICS AND MECHANISM, by A. A. Frost and R. G. Pearson. Second edition, 405 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1961. Price, $11.00.

STATICS, by Horace Lamb. Third edition, reprinted, 357 pages, diagrams, 5½ × 8¼ in. New York, Cambridge University Press, 1961. Price, $3.75 (paper).

ROCKET DEVELOPMENT, by Robert H. Goddard. Reprinted from the *American Legion Magazine*, 222 pages, plates, 5½ × 8 in. Englewood Cliffs, Prentice-Hall, Inc., 1961. Price: $3.95 (cloth); $2.45 (paper).

WATER THE MIRROR OF SCIENCE, by Kenneth S. Davis and John Arthur Day. 195 pages, plates, diagrams, 4¼ × 7¼ in. New York, Doubleday & Company, 1961. Available to secondary school students and teachers through Wesleyan University Press. Price, $0.95 (paper).

THE NATURE OF VIOLENT STORMS, by Louis J. Battan. 158 pages, diagrams, plates, 4¼ × 7¼ in. New York, Doubleday & Company, 1961. Available to secondary school students and teachers through Wesleyan University Press. Price, $0.95 (paper).

CURRENT PROJECTS ON ECONOMIC AND SOCIAL IMPLICATIONS OF SCIENTIFIC RESEARCH AND DEVELOPMENT 1960, compiled by Theodore Suranyi-Unger, Jr., assisted by Elizabeth Harris. 124 pages, 6 × 9 in. National Science Foundation publication NSF-60-79. Washington, Government Printing Office, 1961. No price.

RARE METALS HANDBOOK, edited by Clifford A. Hampel. Second edition, revised, 715 pages, illustrations, 6¾ × 10 in. New York, Reinhold Publishing Corp., 1961. Price, $20.00.

SPECTROCHEMICAL ANALYSIS, by L. H. Ahrens and S. R. Taylor. Second edition, revised, 454 pages, diagrams, 6 × 9 in. Reading (Mass.), Addison-Wesley Publishing Co., Inc., 1961. Price, $15.00.

BOOK NOTES

HEATHLAND ECOLOGY, by C. P. Friedlander. 87 pages, illustrations, plates, $7\frac{1}{4} \times 4\frac{3}{4}$ in. Cambridge, Harvard University Press, 1961. Price, $1.75.

General principles of ecology are presented as they are revealed by a study of heathland. The interdependence of plants and animals and their dependence and effect on the soil are discussed. The author has chosen the heathland as the basis of his text, since it is a comparatively easy place to begin the study of ecology. An introductory chapter gives the characteristics of heathland; the three remaining chapters deal with soil analysis, botany and zoology of the heathland. A section for teachers gives suggestions for class practical work, and includes a "diary" of the heathland month by month. The book is an attractive addition to the Scholarship Series in Biology.

SOURCE BOOK OF THE NEW PLASTICS, Vol. 2, edited by Herbert Simonds. 310 pages, illustrations, 6 × 9 in. New York, Reinhold Publishing Corp., 1961. Price, $8.95.

A continuation of the editor's 1959 Vol. 1, this source book covers new materials and processes through most of 1960. Properties, price, production, application and selection are given for all new important plastics. The discussion on technological progress includes stereospecific catalysts, block and graft polymers, fluorocarbon nomenclature, Montecatini developments and urethane foam progress. In addition to new materials, improvements are listed for 39 established plastics materials. This authentic progress report will serve the plastics industry well.

AERO-SPACE MEDICINE, edited by Harry G. Armstrong. 633 pages, illustrations, $6\frac{3}{4} \times 10$ in. Baltimore, Williams & Wilkins Company, 1961. Price, $18.00. (Exclusive agents in U. S.)

With the tremendous spurt in aviation and space, it is obviously impossible for one to write a book about all the facets of this science. Thus a book with the all-encompassing title of *Aero-Space Medicine* must of necessity be written by a group of men, all specialists in their own fields. Accordingly, 21 contributors have been brought together under the direction of Maj. Gen. Harry G. Armstrong to put together a definitive work on the medical problems of both aviation and space.

Gen. Armstrong indicates that a knowledge of aero-space medicine must be acquired by three different groups. The first is the student and practitioner of aero-space medicine. It is the editor's hope that this volume will contain the basic information on the subject so that the student may use this as a textbook. It is indicated that the flight surgeon will also find this a handy and useful reference. The second group contains those practitioners whose work demands a knowledge of aero-space medicine. Because of the information explosion in the past few years, no longer are there up-to-date books which can serve these doctors. The third group who will find this book of value are the engineers charged with the design, construction and operation of vehicles—either in air or space. To this last group the requirements for properly designed equipment are spelled out.

Aero-Space Medicine, while appealing to a small segment of the professional groups, is still a welcome addition to our bookshelves. With the tremendous emphasis on space, we finally have collected in a single volume the basic material to help those whose work takes them into this new field.

ESSENTIALS OF DIELECTROMAGNETIC ENGINEERING, by H. M. Schlicke. 242 pages, diagrams, $5\frac{1}{4} \times 9$ in. New York, John Wiley & Sons, Inc., 1961. Price, $9.50.

The equivalent circuits and models introduced in this text will appeal to engineering students and educators. Dr. Schlicke wrote this in the language and concepts of the electronics engineer and intended it as an introductory text of graduate level. The reader

will find teachings of the scientific approach in solving practical problems of modern electronic ceramics.

Those looking for facts and stimuli about the possibilities and limits inherent in magnetically soft ferrites and high-permittivity dielectrics, like barium titanates, which play a fascinating and ever increasing role in modern electronic circuits, will find a systematic work of paramount interest.

In his preface, the author states that he has described only a few typical materials and their criteria, presenting those best suited to the various categories of applications, in order to avoid early obsolescence of his text which covers still rapidly expanding technological areas.

To give a brief outline, Chapter I examines relations in electric, magnetic, and electromagnetic fields in materials with *large*, but *constant*, material parameters. Chapter II portrays the frequency, amplitude, and temperature dependencies of ferrites and high-permittivity dielectrics, demonstrating how much actual dielectromagnetic ceramics deviate from the simplifying assumptions made in Chapter I. Chapter III treats lumped reactive circuit elements in conventional and unconventional uses, and Chapter IV deals with the elimination and utilization of macroscopic electromagnetic resonance phenomena. The final chapter deals with a few selected classes of unique applications—nonlinearity, nonreciprocity, and utilization of losses.

FIELD THEORY FOR ENGINEERS, by Parry Moon and Domina Eberle Spencer. 530 pages, diagrams, 6 × 9 in. Princeton, D. Van Nostrand Co., Inc., 1961. Price, $12.75.

The authors, whose many papers in this JOURNAL are referred to in this new work, apply boundary-value techniques to engineering problems. Using a unified field theory approach, the authors treat electromagnetic, acoustical, thermal and gravitational fields. The principal technique employed in the book is that of the separation of variables. This is the first reference volume and text on field theory for engineers; a knowledge of differential equations, vector analysis and elementary physics is assumed. Two features of interest are the unusual treatment of running heads (each right-hand page indicates the subject matter treated thereon) and the grouping of chapter references by topics. Each chapter ends with a group of problems specially aimed at engineers.

ELECTRODYNAMICS OF CONTINUOUS MEDIA, by L. D. Landau and E. M. Lifshitz, translated from the Russian by J. B. Sykes and J. S. Bell. 417 pages, diagrams, 6 × 9 in. London, Pergamon Press; Reading (Mass.), Addison-Wesley Publishing Co., Inc.; 1960. Price, $12.50 (U. S. and Canada only, from Addison-Wesley).

Volume 8 of the *Course in Theoretical Physics* treats the theory of electromagnetic fields in matter and the macroscopic electric and magnetic properties of matter. English-speaking workers will welcome this translation of another volume of this important series of Russian works, by two eminent Russian scientists. Fifteen chapters cover such topics as electrostatics of dielectrics (including thermodynamic relations), constant magnetic field, ferromagnetism, magnetic fluid dynamics, anisotropic media, scattering, and diffraction.

ADVANCES IN ELECTRONICS AND ELECTRON PHYSICS, VOL. XIII, edited by L. Marton. 436 pages, diagrams, plates, 5¾ × 9 in. New York, Academic Press Inc., 1960. Price, $13.50.

Volume XIII in this series contains five papers on diverse phases of electron physics. Two authors are from England, two from the U. S., and one from France. "Inelastic Collisions between Atomic Systems," by J. B. Hasted, discusses classification of collisions, experimental methods of studying inelastic collisions, and collision cross sections. Erwin W. Müller, in "Field Ionization and Field Ion Microscopy," covers field ionization of free atoms and the effects of a metal surface on fields and ion emission. "Velocity Distribution in Electron Streams," by P. A. Lindsay, discusses thermionic cathodes, plane systems, cylindrical systems and magnetic fields in relation to velocity of electron streams. "Electron Probe Analysis," by Raymond Castaing, deals with fundamentals of qualitative analysis by x-ray emission; it also discusses the general structure of the microanalyzer and touches on the contribu-

tion of its use to scientific research. The final paper, by Paul K. Weimer, is a research review of television camera tubes and their components.

THEORY OF FORMAL SYSTEMS, by Raymond M. Smullyan. 126 pages, diagrams, 7 × 10 in. Princeton, Princeton University Press, 1961. Price, $3.00 (paper).

Combining an introduction to recursive function theory with a discussion of new results in the field, the author has put together an interesting study for mathematicians who have no background in mathematical logic. The five chapters cover: formal mathematical systems; formal representability and recursive enumerability; incompleteness and undecidability; recursive function theory; and creativity and effective inseparability. A fifteen-page supplement sketches briefly the structures of systems whose logical bases include the first order predicate calculus, and discusses their relationship to the main chapters in the book.

KINEMATICS AND LINKAGE DESIGN, by Allen S. Hall, Jr. 162 pages, diagrams, 7 × 9¼ in. Englewood Cliffs, Prentice-Hall, Inc., 1961. Price, $6.35.

The basic techniques of plane linkage design are presented in concise and attractive form, with the main emphasis placed on the four-bar linkage. Effective use is made of the properties of the polhode (or wave of instantaneous center) in the synthesis of mechanisms; other methods include both preliminary graphic and more exact analytical techniques. The final chapter describes several ways of designing a linkage for guiding a body through a finite number of specified points. The many diagrams are clear, and economically presented, and there are many references to current literature, as well as numerous examples, of which each chapter contains several that are fully worked out to illustrate the theory.

LIQUID ROCKETS AND PROPELLANTS, Vol. II of Progress in Astronautics and Rocketry, edited by Loren E. Bollinger, Martin Goldsmith and Alexis W. Lemmon, Jr. 682 pages, diagrams, 6 × 9¼ in. New York, Academic Press Inc., 1960. Price, $6.50.

Six sections (28 papers) comprise this second volume in the Progress in Astronautics and Rocketry series sponsored by the American Rocket Society. The first section covers simulated high-altitude testing of rockets, with four papers on diffusers and one on an ejector system. Instrumentation for combustion instability is dealt with in the five papers in the second section. The three papers in the third section are on liquid propellant combustion and treat monodisperse spray deflagration, an interferometric technique for the study of fuel combustion, and single-droplet burning. Six papers (the fourth section) discuss various aspects of reaction kinetics, including hydrazine decomposition, spectroscopic techniques, liquid ozone, and detonability of ozone and nitric oxide. Following these four sections on basic research problems, the two final sections are devoted to propellant selection (2 papers) and rocket engine design (7 papers).

SCIENTIFIC THINKING AND SCIENTIFIC WRITING, by Martin S. Peterson. 205 pages, diagrams, illustrations, 5¾ × 9 in. New York, Reinhold Publishing Corp., 1961. Price, $6.95.

This is one of the best books this reviewer has seen on how to prepare a scientific paper. Instead of emphasizing the mechanics of style, punctuation, etc. (the editor's province), the author presents valuable suggestions for writing scientific articles in accordance with the logic of science. In his preface, he states "Tracing the genesis of an original scientific concept from the intuitive or concept stage on through experimentation to the discovered or verified generalization is mental action that produces the text for most journal articles—and if the tracing wavers at some points or skips over some of the ground covered, the reader has to do the work. Many readers refuse to carry the author's burden for him, and lay the article aside unfinished." To assure readership, authors of scientific articles should set forth the *why*, the *how*, and the *what* of their findings, in a logical form. This book, written in accordance with the author's suggestions to others, should improve the quality of expository writing in scientific fields—if its advice is heeded. It is recommended to all scientists and engineers who must, in the

course of their work, prepare journal articles, review articles, official progress reports or critical book reviews. There are valuable suggestions and specific ideas included for all these types of scientific writing.

ELECTRONICS, by Paul M. Chirlian and Armen H. Zemanian. 327 pages, diagrams, 5¾ × 9 in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $8.75.

The authors, both associate professors of electrical engineering, intended their work as an undergraduate college textbook on the physical basis of electronic devices and their circuit representations. They present a clearly understood discussion of the physical phenomena occurring in the common electronic devices, such as transistors, gas tubes, vacuum tubes, and tunnel diodes, and emphasize the similarities in these pheonomena wherever possible.

They lay the foundation for the analysis of electronic circuits by developing the equivalent circuits of the devices and illustrating their use, and the analysis is developed in a consolidated fashion so that the reader will appreciate the common features of these devices without overlooking their essential differences.

A hypothetical 3-terminal device, a "generalized device," is introduced by the authors. Using this device, whose input or controlling parameter may represent either a current or a voltage or any variable, the development is immediately applicable to many electronic devices thus emphasizing the similarities in the circuit applications.

PHYSICS AND ARCHAEOLOGY, by M. J. Aitken. 171 pages, diagrams, plates, 5¾ × 9 in. New York, Interscience Publishers, Inc., 1961. Price, $6.00.

This text describes some of the ways in which physics has been applied to archaeology and was written for two groups of readers. The general scientific reader will find it interesting, and it is of value for the student of Archaeometry, which is measurements made on archaeological material.

The author has omitted any overlapping of chapters, so that a reader may exclude a part he finds too technical without fear of losing his track, and has included an abundance of references to original papers for those wishing to pursue things further. The emphasis of the book is on methods which produce objective, quantitative results. There are eight chapters: 1. Finding; 2. Magnetic Location; 3. The Proton Magnetometer; 4. Resistivity Surveying; 5. Dating; 6. Radiocarbon Dating; 7. Magnetic Dating; 8. Analysis.

CERAMICS, by P. William Lee. 205 pages, illustrations, 5⅞ × 9 in. New York, Reinhold Publishing Corp., 1961. Price, $5.95.

Both technical and non-technical readers who wish to acquire an all-around knowledge of ceramics will find this book very useful, as it is the first work to present a comprehensive survey of the entire field of ceramics within a single volume. Even ceramic industries will find this text a compact guide and reference to ceramics as an industrial material.

The applications of ceramic materials in industry, their history, raw materials and basic chemistry are covered, along with such specialized topics as the use of ceramics in electronics, dentistry, high temperature refractories, and low temperature enamels. New and recent developments in this important field are given special attention by the author.

QUANTUM MECHANICS, by John L. Powell and Bernd Crasemann. 495 pages, 6 × 9 in. Reading (Mass.), Addison-Wesley Publishing Co., Inc., 1961. Price, $9.75.

This is a new text, written on the senior or first-year graduate level. The first five chapters give the historical background, foundations of wave mechanics, wave packets and the uncertainty principle, the Schrödinger equation, and problems in one dimension (including a discussion of parity). The remaining seven chapters, which introduce the formal structure of quantum mechanics, cover operators and eigenfunctions, spherically symmetric systems, theory of elastic scattering, matrix mechanics, angular momentum and spin, perturbation theory and identical particles. The essentially algebraic structure of quantum mechanics is emphasized and symmetry operations are given a prominent role.

PROGRESS IN OPERATIONS RESEARCH, Volume 1, edited by Russell L. Ackoff. 505 pages, diagrams, 6 × 9 in. New York, John Wiley & Sons, Inc., 1961. Price, $11.50.

This is the first of a series of review volumes covering mathematical techniques and methods in operations research. The series is intended to be of a reference nature on major techniques, with articles contributed by experts. Volume 1 is on technical progress in eight fields: inventory theory, linear programming, dynamic programming, queuing theory, sequencing theory, replacement theory, simulation, and gaming. The current state of the art is surveyed in each field. In addition, the first two chapters are an introductory one on operations research methods in general and one on decision and value theory.

THEORIES OF ENGINEERING EXPERIMENTATION, by Hilbert Schenck, Jr. 239 pages, diagrams, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $7.00.

New in approach, this is basically a lab book in which mechanical, test code, and use-instruction details are left entirely to the individual college instructor. It covers the theoretical, analytic, and statistical aspects of basic engineering experimentation, dealing specifically with the statistics of instrument error, the propagation of such error, the selection of instruments based on error-study and the choice of proper settings for test equipment to minimize uncertainty and error.

The use of balance equations, extrapolation, and replication are considered as means of locating uncertainty or error in existing data. The rejection of outliers or "wild points" is explained, and the author discusses dimensional analysis, test point spacing, and suppression of extraneous variable effects using Graeco-Latin square test plans. The text is concluded with a discussion of fundamental ways of analyzing test data, with the major subdivision being statistical, graphical, and mathematical in character. Three complete experiments, common to most college-trained engineers, are included in one of the three appendixes to this book.

LECTURES ON COMMUNICATION SYSTEM THEORY, edited by Elie J. Baghdady. 617 pages, diagrams, 6 × 9 in. New York, McGraw-Hill Book Co., Inc., 1961. Price, $12.50.

This is an expansion of a set of lecture notes for a special summer course given at MIT in 1959 on "Reliable Long-Range Radio Communications." The volume presents 23 papers by 18 contributors on various phases of modern approaches to the development of new communication systems. The approach is theoretical, with emphasis on the use of basic models and techniques in the design and evaluation of communication systems. Four broad fields are covered: (1) mathematical representation of signals and disturbances (3 papers); (2) characterization of transmission channels (4 papers); (3) corrective measures for disturbances in the transmission channel (10 papers); and (4) synthesis of communication systems (6 papers). Many important unsolved research programs are pointed out, which should make the book of value as a text as well as a reference.

ELEKTRONISCHE HILFSMITTEL DES PHYSIKERS, by Wolfgang Gruhle. 200 pages, diagrams, 5½ × 8 in. Berlin-Springer-Verlag, 1960. Price, DM 29.60.

This book lives up to its title, "Electronic Aids for the Physicist," for it presents methods for solutions to practical problems encountered by experimental physicists. Profusely illustrated with circuit diagrams, the text is a valuable time-saver for those who are interested in developing electronic devices. Each chapter is self-contained. In addition to a summary chapter, there are six chapters covering: (1) basic elements such as tubes, semiconductors, and RC-RL networks; (2) signalling (generators, multivibrators, etc.); (3) signal change, both linear and nonlinear; (4) networks, frequency change (both parallel and series), and inspection techniques; (5) signal recording, both analog and digital; and (6) circuits and high tension systems for power distribution.

NEGATIVE ION STUDIES

The National Bureau of Standards has been conducting studies of the photodetachment of electrons from negative atomic and molecular ions[1] in an investigation sponsored by the Office of Naval Research. Direct interest has centered on the cross sections of negative atomic hydrogen ions and negative atomic and molecular oxygen ions because of their geophysical and astrophysical importance, although other ions having significant applications have been studied.

Theories of the ionosphere must take into account the possible diurnal variations in electron and ion densities, which may depend on the competition between the formation of negative ions by attachment of electrons and the destruction of negative ions by photodetachment. The processes of formation and destruction of negative ions must also be considered in any thermodynamic treatment of systems involving very hot gases, for electrons can become attached to some of these gases to form negative ions. For example, solar emissivity in the visible spectrum is controlled by the hydrogen ion through bound-free transitions.

Information obtained on the continuous solar spectrum in the region between 0.6 and 1.6 microns shows departures from the Planck curve characteristic of a black body. These departures were first ascribed, by R. Wildt at Yale in 1939, to the continuous absorption of the negative hydrogen ions in the solar photosphere.[2] In 1950, W. L. Fite at Harvard attempted to detect photodetachment from the negative hydrogen ion. Although his experiment was not successful, it led to the development of the high-current negative ion source now in use at the Bureau and helped determine the present experimental approach to the problem.

The experimental study of photodetachment of electrons from negative ions, a special case of photoelectric absorption, originated in 1953[3] when L. M. Branscomb, of the Bureau staff, and W. L. Fite, a consultant to the Bureau, made the first laboratory observation of the process

$$H^- + h\nu \rightarrow H + e.$$

[1] L. M. Branscomb and S. J. Smith, *Phys. Rev.*, Vol. 98, No. 4, p. 1028 (1955); S. J. Smith and D. S. Burch, *Phys. Rev. Letters*, Vol. 2, No. 4, p. 165 (1959); S. J. Smith and D. S. Burch, *Phys. Rev.*, Vol. 116, No. 5, p. 1125 (1959).

[2] R. Wildt, *Astrophys. J.*, Vol. 89, p. 295 (1939).

[3] L. M. Branscomb and W. L. Fite, *Phys. Rev.*, Vol. 93, p. 651 A (1954).

A small current of free electrons produced in high vacuum by photon absorption at the intersection of two mutually perpendicular beams, one of negative ions and the other of intense visible light, was collected and measured. In 1954 and 1955, measurements of photodetachment cross sections for negative ions of several gases[4] were made at the Bureau by L. M. Branscomb and S. J. Smith using the same crossed-beam geometry and very simple ion and photon sources and optics.[5]

This high-vacuum crossed-beam apparatus, developed by the Bureau for the study of photodetachment of electrons from negative ions, has recently been improved.[6] The refinements by Smith and Branscomb have made possible a very accurate measurement of the electron affinity of atomic oxygen[7] and reliable point-by-point measurements of the spectral dependencies of the photodetachment cross sections for negative atomic hydrogen ions and negative atomic and molecular oxygen ions.[8]

The improvement of this crossed-beam apparatus has included the development of (1) an optical system which permits the use of sets of filters to obtain quasi-monochromatic light; (2) a mass spectrometer having a high-transmission magnetic sector field to ensure the purity of the ion beam; and (3) an a-c. preamplifier, amplifier, and synchronous detector which permit detection of the photodetached electrons with very high sensitivity. The choice of the instrumentation involved a difficult search for a workable compromise in meeting simultaneous requirements for resolution and intensity in both the ion beam and the photon beam, under severe limitations imposed by available detection sensitivity.

A single ion passing through the photon beam has its extra electron photodetached with a probability dependent upon the photodetachment cross section and the photon flux density along the path of the ion. Cross sections successfully measured have ranged from about 4×10^{-17} cm^2 at the maximum for H^- down to about 5×10^{-20} cm^2 near the threshold for O_2^-. These limits correspond to photodetachment probabilities ranging from about 10^{-5} to about 1.5×10^{-7} per watt/cm^2 per cm of ion path, assuming ion velocities corresponding to 300 ev of energy. Actual photon beam intensities were of the order of one-quarter volt.

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[4] L. M. Branscomb and S. J. Smith, *Phys. Rev.*, Vol. 98, p. 1028 (1955); *Phys. Rev.*, Vol. 98, p. 1127 (1955).

[5] S. J. Smith and L. M. Branscomb, *J. Research NBS*, Vol. 55, p. 165 (1955).

[6] For further information, see "Optical Methods for Negative Ion Studies," by Stephen J. Smith and Lewis M. Branscomb, *Rev. Sci. Inst.*, Vol. 31, No. 7, pp. 733–747 (1960).

[7] L. M. Branscomb, D. S. Burch, S. J. Smith and S. Geltman, *Phys. Rev.*, Vol. 111, p. 504 (1958).

[8] S. J. Smith, *Proc. Fourth Internatl. Conf. Ion. Phenom. in Gases;* D. S. Burch, S. J. Smith, and L. M. Branscomb, *Phys. Rev.*, Vol. 112, p. 171 (1958); S. J. Smith and D. S. Burch, *Phys. Rev. Letters*, Vol. 2, p. 165 (1959); *Phys. Rev.*, Vol. 116, p. 1125 (1959).

The photodetachment process is in competition with the detachment of electrons by collisions with the background gas. The Bureau's experiments were conducted at pressures of the order of 10^{-6} mm Hg, for which the collisional detachment probability is roughly 2×10^{-5} per cm of path,[9] yielding a current of free electrons which is larger than that produced by photodetachment. The photodetachment signal is separated from the collisional detachment background by modulation of the photon beam at 450 cps and measurement of that component of the free electron current which alternates synchronously with the modulated photon beam. Ideally, the photodetached electron current yields an a-c. signal having the same waveform as the chopped light, while the collisionally detached electron current yields a d-c. signal. A measurement of the a-c. component of the detached electron current using appropriate filter networks to eliminate the d-c. component is therefore a measurement of the photodetachment current.

The electrons detached in the reaction chamber at the crossing of ion and photon beams are separated from the ions and trapped by weak magnetic and electric fields. They are then collected on a metal cap which connects directly into the grid circuit of a very-high-impedance preamplifier.

The ions studied thus far by the photodetachment method are H^-, O^-, O_2^-, OH^-, OD^-, and S^-;[10] which, in addition to possessing interesting applications, can be produced relatively easily as anode rays from a low-pressure discharge in the quantities required. In addition, they are low-mass ions, which simplifies the problem of mass resolution. The study of less easily produced and analyzed ions is the next objective of the photodetachment program.

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[9] J. B. Hasted, *Proc. Roy. Soc. (London)*, Vol. A212, p. 235 (1952).

[10] See footnotes 2–5; also, L. M. Branscomb and S. J. Smith, *J. Chem. Phys.*, Vol. 25, p. 598 (1956); S. J. Smith and L. M. Branscomb, *Bull. Am. Phys. Soc. Ser. I*, 30, 31 (1955).

THE FRANKLIN INSTITUTE

MINUTES OF THE STATED MEETING

May 17, 1961

The Stated Meeting of The Franklin Institute was held at 8:15 P.M. in the Lecture Hall, Wynn Laurence LePage, President, presiding. Mr. LePage called the meeting to order with approximately 150 members and guests present.

The President stated that the minutes of the Stated Meetings of the Institute of February 15, 1961, and March 15, 1961, were published in the March and April issues of the JOURNAL, respectively. There being no corrections or additions, the minutes were approved as published.

Mr. LePage announced that tonight's lecture will be the Eighth Edward G. Budd Lecture. This Lecture was established in 1948 by the family of the late Edward G. Budd, as a tribute to his long-time membership in The Institute and its Board of Managers on which he had served from 1927 until his death in 1946.

The President then introduced the speaker of the evening, Welcome W. Bender, Director of the Research Institute for Advanced Study, The Martin Company.

Mr. Bender's talk on "Pure Science—Stepchild of Economy" brought out that some observers feel that limited support through philanthropic channels is causing basic research to lag in supplying the needs of applied science and engineering. He discussed the new concept, as provided by the Research Institute for Advanced Study, of a pure science center which is considered as an investment.

After an interesting discussion period, Mr. LePage adjourned the meeting at 9:35 P.M. with sincere thanks to Mr. Bender for his thought-provoking talk.

WILLIAM F. JACKSON, JR.
Secretary

COMMITTEE ON SCIENCE AND THE ARTS

(Abstract of Proceedings of Stated Meeting held Wednesday, May 10, 1961)

HALL OF THE COMMITTEE,
PHILADELPHIA, MAY 10, 1961

Mr. FRANCIS G. TATNALL in the Chair.

The following reports were presented for final action:

No. 3312: Chemical Regulation of Plant Growth.

This report recommended the award of a John Price Wetherill Medal each to Percy W. Zimmerman and Albert E. Hitchcock, of Yonkers, New York, "In consideration of their researches on the chemical factors controlling the growth of plants, which have found application in agriculture."

No. 3394: Clamer Medal.

This report recommended the award of the Francis J. Clamer Medal to Russell P. Heuer, of Bryn Mawr, Pennsylvania, "In consideration of his pioneer work in the composition and manufacture of basic brick and in the design of open-hearth roof structures which have been major contributing factors in materially increasing the rate of steel production from open-hearth furnaces."

No. 3395: Brown Medal.

This report recommended the award of the Frank P. Brown Medal to Le Corbusier, Charles-Edouard Jeanneret, of Paris, France, "For a lifetime of creative leadership in the teaching and the practice of Architecture as exemplified by his modular concept of human scale, free plan, pillar foundations, glass walls, sun breaks and roof terraces; his City Planning as exemplified by the Vertical City concept and for his outstanding contributions as a sculptor and painter."

D. S. FAHRNEY,
Secretary to Committee

NEW BOOKS IN THE FRANKLIN INSTITUTE LIBRARY

CHEMISTRY AND CHEMICAL ENGINEERING

Detergents, by G. Carrière. Elsevier, 1960.
Destillation im Laboratorium, by H. Röck. Steinkopff, 1960.
Process Characterization, by H. I. Waterman. Elsevier, 1960.
Activation Analysis Handbook, by R. C. Koch. Academic, 1960.
Size and Shape Changes of Contractile Polymers, edited by A. Wassermann. Pergamon, 1960.
Methods in Geochemistry, edited by A. A. Smales. Interscience, 1960.
Diffusion in Solids, Liquids, Gases, by W. Jost. Academic, 1960.
Cobalt, by R. S. Young. Reinhold, 1961.
Engineering Design with Rubber, by A. R. Payne. Interscience, 1960.
Polypropylene, by T. O. J. Kresser. Reinhold, 1960.
The Technique of Photomicrography, by D. F. Lawson. Macmillan, 1960.
Preparative Methods of Polymer Chemistry, W. R. Sorenson. Interscience, 1961.
Manual of Man-Made Fibres, by C. Z. Carroll-Porczynski. Chemical Pub. Co., 1961.

ELECTRICAL ENGINEERING

Glossary of Terms in Computers and Data Processing, by E. C. Berkeley. 5th ed. Berkeley Enterprises, 1960.
Electrical Precipitation. Pennsylvania State University, 1960.
Logique Générale des Systèmes et des Effets, by A. Ducrocq. Dunod, 1960.
Werkstoffe Für Elektrische Kontakte, by A. Keil. Springer, 1960.
Integrated Data Processing and Computers, report on a mission to the United States by a group of European experts. Paris, 1960.
Electronic Business Machines, by J. H. Leveson. Philosophical Library, 1960.
Radio Transmitters, by L. Gray. McGraw-Hill, 1961.
Wave Generation and Shaping, by L. Stauss. McGraw-Hill, 1960.
International Conference on Medical Electronics, edited by C. N. Smyth. Iliffe, 1960.
Linear Circuits, by R. E. Scott. Addison-Wesley, 1960.
Analytical Techniques for Non-Linear Control Systems, by J. C. West. Van Nostrand, 1960.
High-Frequency Magnetic Materials, by W. J. Polydoroff. Wiley, 1960.
Analogue Computation, by S. Fifer. McGraw-Hill, 1961.
Electronic Equipment Reliability, by G. W. A. Dummer. Wiley, 1960.
High Frequency Applications of Ferrites, by J. Roberts. Van Nostrand, 1960.
Linear Systems Analysis, by P. E. Pfeiffer. McGraw-Hill, 1961.

HISTORY OF SCIENCE AND TECHNOLOGY

The Institution of Naval Architects, 1860–1960, by K. C. Barnaby. Royal Institution of Naval Architects, 1960.
Three Essays: A Dissertation on the Genesis of Mechanical Transport in America Before 1800, by G. Bathe. 1960.
The Life and Work of William Higgins, by T. S. Wheeler. Pergamon, 1960.
George and Robert Stephenson, by L. T. C. Rolt. Longmans, 1960.
A Guide to American Trade Catalogs, 1744–1900, by L. B. Romaine. Bowker, 1960.
C. F. Gauss, by H. Reichardt. Haude & Spenersche Verlagshichhandlung, 1960.

MATHEMATICS

Time Harmonic Electromagnetic Fields, by R. F. Harrington. McGraw-Hill, 1961.
Linear Integral Equations, by S. G. Mikhlin. Hindustan Pub. Corp., 1960.
Integral Quadratic Forms, by G. L. Watson. Cambridge University Press, 1960.

MATHEMATICAL HANDBOOK FOR SCIENTISTS AND ENGINEERS, by G. A. Korn. McGraw-Hill, 1961.

DYNAMIC PROGRAMMING AND MARKOV PROCESSES, by R. A. Howard. Technology Press of Massachusetts Institute of Technology, 1960.

PHYSICS

NEKOTORYE ZADACHI TEORII UPRUGOSTI I TEPLOPROVODNOSTI, by B. G. Korenev.

AXIOMATICS OF CLASSICAL STATISTICAL MECHANICS, by R. Kurth. Pergamon, 1960.

BRITISH COAL UTILIZATION RESEARCH ASSOCIATION. AERODYNAMIC CAPTURE OF PARTICLES, edited by E. G. Richardson. Pergamon, 1960.

EINFÜHRUNG IN DIE KORNGRÖSSENMESSTECHNIK, by W. Batel. Springer, 1960.

THE FERMI SURFACE, edited by W. A. Harrison. Wiley, 1960.

TRANSPORT PHENOMENA, by R. B. Bird. Wiley, 1960.

MECHANIK DER FESTEN KÖRPER, by H. Parkus. Springer, 1960.

SCUOLA INTERNAZIONALE DI FISICA "ENRICO FERMI." CORSO XIII: FISICA DEL PLASMA. N. Zanichelli, 1960.

SCUOLA INTERNAZIONALE DI FISICA "ENRICO FERMI." CORSO X: TERMODINAMICA DEI PROCESSI IRREVERSIBILI. N. Zanichelli, 1960.

EINFÜHRUNG IN DIE THEORETISCHE GASDYNAMIK, by R. Sauer. Springer, 1960.

MATHEMATISCHE ELASTIZITATSTHEORIE DER EBENEN PROBLEME, by I. Babuska. Akademie-Verlag, 1960.

ELEMENTARE SCHALENSTATIK, by A. Pflüger. Springer, 1960.

MECHANIK DEFORMIERBARER KÖRPER, by M. Päsler. Gruyter, 1960.

INTRODUCTION TO THE THEORY OF IONIZED GASES, by J. L. Delcroix. Interscience, 1960.

ELECTROMAGNETIC FIELDS AND WAVES, by R. V. Langmuir. McGraw-Hill, 1961.

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MECHANICAL WAVEGUIDES, by M. Redwood. Pergamon, 1960.

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PRESIDENT'S LETTER PRESENTING THE ANNUAL REPORT FOR 1960 *

The year 1960 was the 136th consecutive year during which The Franklin Institute has fulfilled its dedication to the "promotion of the mechanic arts" in the City of Philadelphia. It was a year in which more persons passed through its Museum and Planetarium than ever before, 439,986 to be exact, and it was a year in which the Institute's educational facilities provided supplementary Museum teaching to more school children on more diverse subjects than ever before in its history. This Annual Report is submitted to the membership so that each of you may become more familiar with the manner in which this unique institution serves the wide region in which is located the City of Philadelphia.

Following this letter you will find a factual account of the activities during 1960 of each of The Franklin Institute's eleven departments. I would be remiss, however, if I failed to point out certain high spots for the year.

An outstanding and unique chemistry exhibit was opened in the Science Museum. The completeness and remarkable characteristics of this exhibit will, we expect, set the pace for future Museum educational exhibits both here and, indeed, throughout the country for some years to come. Not only is its concept unique in the breadth of the science that it covers, but its authenticity was assured by a united effort involving dedicated volunteers from national technical societies, the exhibit builders and our Museum staff—a cooperative effort of remarkable effectiveness.

The Institute's already interesting Meetings Program was, we believe, made more effective during the year by the addition of speakers on subjects of more popular appeal than in the past, without sacrifice in calibre.

The Franklin Institute Laboratories for Research and Development have completed a year of transition. Intensive efforts were made further to entrench the Laboratories in the disciplines in which lie their greatest skills, and progress was made in several new disciplines. Lines of responsibility in the Laboratories' organization were more precisely defined. I believe we are now better prepared for an expanded volume of research than we have been for some time in the past. In terms of research contracts received, 1960 was the second best year in the Laboratories' history.

The Computing Center went through a most critical year. However, under its new Director, Laurence C. McGinn, a basic realignment was made, both in personnel and space occupied. By year's end, operation of the Computing Center had changed from a monthly loss to a surplus. Furthermore, its educational activities were considerably broadened, and I now look to the future of this activity with far greater optimism than I have in the past.

The Bartol Research Foundation of The Franklin Institute moved smoothly from the administration of Dr. W. F. G. Swann, now Director Emeritus, to that

* A copy of the complete Annual Report is available on request to the Public Relations Department of The Franklin Institute, 20th and the Parkway, Philadelphia 3, Pa.

of Dr. Martin A. Pomerantz, Director. Research activity in the field of low-energy physics reached a new high bracket in quality as well as quantity, and Bartol's pre-eminence in cosmic radiation research and allied outer space investigation has been consistently furthered.

Under the direction of Dr. William G. Batt, the Biochemical Research Foundation at Newark, Del., for which The Franklin Institute serves as Trustee, continued its progress in cancer research and the fundamental investigation of cell metabolism. Important strides were made in each of these subjects.

The Franklin Institute Library, under its new Director, John P. McGowan, has greatly increased its usefulness through the complete reorganization of its resources, as a result of which and through formal action by the Board of Managers, this outstanding technical collection has, for the first time, become a true public resource. Last, and far from least, our membership body stands at a new high of 7,579 persons who, I believe, are dedicated to making this great institution the science education and scientific research headquarters for the entire Delaware Valley area.

During the year two principal staff changes were made. On February 1, 1960, J. G. Richard Heckscher was appointed Executive Vice President of The Franklin Institute, and in this capacity has complete charge of operations. On December 31, 1960, we regretfully accepted the resignation of Dr. Nicol H. Smith, who had served the Institute faithfully for twenty-eight years, and who relinquished the Directorship of The Franklin Institute Laboratories, which he had directed since 1949. We wish Dr. Smith a happy and fruitful retirement.

The year 1960 saw two events which, I believe, will become increasingly significant as the years go by. Following closely on the heels of President Eisenhower's declaration of concern over the shortage of trained scientists and engineers in the U. S. and the inevitable crisis toward which the nation was heading if steps to alleviate this situation were not taken, The Franklin Institute, after study and appraisal, announced a ten year "Program to Advance Science Education." The Program, when in full operation in late 1964, concerted with the Institute's present broad activities in the field of science education, will be making real inroads into the potential scarcity of technically trained persons by inspiring children, at the real age of determination of future careers, inspiring them to deep interest in science and technology and leading them to the colleges and universities of scientific learning. There is little doubt, too, that the program will become a pattern to be followed by other organizations across the country and thus the significance of this important expansion of Institute activity will become national.

The second event of the year headed for increasing significance as time goes by was the successful negotiation of all preliminary arrangements whereby a modern research laboratory will be erected adjacent to the Institute's main edifice, to house the Laboratories for Research and Development. This project, the result of some years of planning on the part of the Board of Managers, is expected to be completed within two years and will be the outcome of the combined cooperation, with the Institute, of the City of Philadelphia and its several specific departments, and the Philadelphia Industrial Development Corporation who will cause the facility to be built without cost to the Institute and made available upon a lease basis.

It goes without saying that none of these accomplishments nor indeed the laying of these plans could be so much as contemplated except with the active help of a dedicated staff. You may feel proud of the contribution which each of the Institute employees has made not only toward furthering science education for the children of the Greater Philadelphia Area, but more and more through our research work toward the end that the Institute may become a center for science activities throughout the United States and abroad.

Respectfully submitted,
By order of the Board of Managers,
WYNN LAURENCE LEPAGE
President

NOTE: On the following pages a summary of the financial statements for 1960 as compared with 1959 is published.

The financial statements in greater detail, together with explanatory notes and the report of our certified public accountants, Lybrand, Ross Bros. & Montgomery, are available at the Treasurer's Office for inspection by members.

THE FRANKLIN INSTITUTE OF THE STATE OF PENNSYLVANIA

BALANCE SHEETS, DECEMBER

Assets

	1960	*1959*
General Operating Fund		
Cash	$ 347,535	$ 643,468
Receivable, including unbilled costs, research contracts	1,075,028	964,531
Other receivables	121,935	159,405
Inventories and work in process	53,535	72,871
Prepaid expenses	22,744	23,991
Total general operating funds	$ 1,620,777	$ 1,864,266
Bartol Research Foundation Operating Fund		
Cash	201,939	251,813
Accounts receivable	64,273	64,271
Inventories and work in process	27,030	21,269
Prepaid expenses	3,478	5,006
Total Bartol operating fund	$ 296,720	$ 342,359
Development Fund		
Cash	9,678	—
Total development fund	$ 9,678	$ —
Endowment and Other Invested Funds		
Investments at cost (market values $8,475,022 and $8,330,540, respectively)	6,162,009	6,135,264
Uninvested principal cash	55,529	34,019
Income cash	47,202	44,114
	$ 6,264,740	$ 6,213,397
DeMonclos Trust Fund		
Investments, at cost	195,576	195,854
Uninvested principal cash	290	12
Income cash	—	2,322
	195,866	198,188
Total endowment funds	$ 6,460,606	$ 6,411,585
General Plant and Property Fund		
Cash	3,000	3,000
Buildings and equipment, net of allowances for depreciation and amortization	4,022,060	3,964,368
Advances to general operating fund	509,632	555,377
Total general plant and property fund	$ 4,534,692	$ 4,522,745
Bartol Research Foundation Plant and Property Fund		
Land, buildings and equipment	$ 892,571	$ 880,994
Total assets (excluding interfund advances)	$13,302,208	$13,463,678

FOR THE PROMOTION OF THE MECHANIC ARTS

31, 1960 AND DECEMBER 31, 1959

Liabilities and Funds

	1960	*1959*
General Operating Fund		
Accounts payable, accrued expenses, etc.	$ 644,821	$ 735,339
Deferred income	49,590	48,478
Advances from plant and property fund	509,632	555,377
Operating fund balance	416,734	525,072
Total general operating fund	$ 1,620,777	$ 1,864,266
Bartol Research Foundation Operating Fund		
Accounts payable and accrued expenses	102,325	182,935
Operating fund balance	194,395	159,424
Total Bartol operating fund	$ 296,720	$ 342,359
Development Fund		
Contributions	27,423	—
Less: Fund raising expenses	(16,124)	—
Allocation to Chemistry Exhibit	(1,621)	—
	$ 9,678	$ —
Endowment and Other Invested Funds		
General and restricted endowment funds and funds functioning as endowment including those of the Bartol Research Foundation	6,217,537	6,169,283
Unexpended income	47,203	44,114
	$ 6,264,740	$ 6,213,397
DeMonclos Trust Fund		
Principal	195,866	195,866
Undistributed income	—	2,322
	195,866	198,188
Total endowment funds	$ 6,460,606	$ 6,411,585
General Plant and Property Fund		
Contributed funds	4,534,692	4,522,745
	$ 4,534,692	$ 4,522,745
Bartol Research Foundation Plant and Property Fund		
Plant and property fund balance	$ 892,571	$ 880,994
Total liabilities and funds (excluding interfund advances)	$13,302,208	$13,463,678

STATEMENTS OF INCOME AND EXPENSES FOR THE YEARS 1960 AND 1959 (UNAUDITED)

GENERAL OPERATING FUND

	1960	1959
Operating Revenue		
Membership dues	$ 75,474	$ 73,372
Museum	270,471	248,949
Laboratory research	3,448,174	3,518,931
Computing Center	338,440	349,151
JOURNAL	30,695	32,037
Other Activities	19,823	40,373
Total operating revenue	$4,183,077	$4,262,813
Operating Costs and Expenses		
Museum	494,689	442,406
Library, net of transfers to other departments	133,643	109,304
Laboratory research	3,394,886	3,357,519
Computing Center	371,594	393,458
JOURNAL	47,544	41,429
Other Activities	92,618	85,142
Administrative and general expenses, net of transfers to other departments	17,717	16,662
Total operating costs and expenses	$4,552,691	$4,445,920
Operating income or (loss) before other income and charges	$ (369,614)	$ (183,107)
Other Income		
From endowment funds	137,366	127,044
Gifts from Friends of Franklin and others	72,673	73,012
Appropriations, state and local	73,735	71,283
Miscellaneous	25,053	24,335
	$ 308,827	$ 295,674
Net income or (loss) before other charges	$ (60,787)	$ 112,567
Other Charges		
Write-off of planning costs of new laboratory building on present grounds, plans for which were abandoned	29,540	—
Miscellaneous	18,010	15,939
	$ 47,550	$ 15,939
Net income or (loss)	$ (108,337)	$ 96,628

THE BARTOL RESEARCH FOUNDATION
(a restricted fund of The Franklin Institute)

Income		
From investments	$ 152,780	$ 140,494
Research projects	376,699	300,915
Other	5,318	4,227
Total income	$ 534,797	$ 445,636
Expenses		
Direct costs		
Research projects	309,815	233,692
Self-sponsored research	37,233	42,859
Other Expenses		
Laboratory	40,377	28,308
Machine Shop	8,169	4,713
Buildings	25,793	25,500
Library	1,989	3,255
Administrative and general	64,182	46,398
Rental properties	692	1,098
	$ 488,250	$ 385,823
Excess of income over expenses	$ 46,547	$ 59,813

YOU CAN ADVANCE SCIENCE EDUCATION

Today, more than ever before in its 137-year history, there is vital need for The Franklin Institute to effectively promote education in science and technology. It is imperative that we meet this challenge by providing adequate educational opportunities in these fields. This requires vision, objective planning, and money. We have more than enough of the first two requisites, but far too little of the third.

Our programs are aimed at professional scientists and industry, as well as the lay public and young people seeking inspiration and guidance in choosing a career. The Institute's educational programs are impressive. They begin with students in the early grades of our elementary schools and continue throughout an individual's professional or industrial life. With more funds at our disposal, the scope and vigor of these activities could be greatly increased and increasingly effective.

The Franklin Institute is not richly endowed. It is a non-profit organization, depending for encouragement and support on an understanding public. Capable and conservative management assures wise administration of all funds.

Your gift or bequest, large or small, will be deeply appreciated and will be used effectively to broaden the Institute's educational usefulness. There is a warm satisfaction in giving financial support to an organization that has pioneered in, and is dedicated to, the advancement of science and technology.

When property is transferred, title should be in the name of The Franklin Institute of the State of Pennsylvania for the Promotion of the Mechanic Arts.

The Secretary of The Franklin Institute will gladly furnish you with additional information. Write to him at The Franklin Institute, Benjamin Franklin Parkway at Twentieth Street, Philadelphia 3, Pennsylvania.

CURRENT TOPICS

Aircraft Anti-Collision System.—A simple and inexpensive radar technique which can be used as an anti-collision device or for highly accurate identification of aircraft in bad weather has been tested by The Martin Company's Orlando Division.

Called "Double Bounce Circular Polarization," the system was described by Sheldon Isaacson, Martin Company engineer, at the international convention of the Institute of Radio Engineers meeting at the New York Coliseum.

The technique overcomes the severe difficulty experienced by conventional radar systems in distinguishing between reflections from targets and from heavy rain.

"Circular polarization" imparts a spin to the radar signal which reverses its direction of rotation each time the signal bounces off an obstacle—much like a billiard ball that has English on it. By making the target signal bounce twice, it can be readily distinguished from the one-bounce reflection from rain because it will be rotating in the opposite direction.

The double bounce is obtained by bouncing the signal from the ground to the target or from one side to the other of a right-angle corner reflector mounted on the target, and then back to the radar receiving antenna.

The technique requires only the addition of an inexpensive antenna horn to conventional radar systems to provide a spin to the radar signal. The placement of simple corner reflectors on the ground along airway paths will provide for positive bad weather navigation guidance for aircraft, while the addition of similar reflectors to the bottom of aircraft will make possible their absolute location and identification by other aircraft or ground control stations.

As an anti-collision device, the system would be usable without the addition of reflectors to aircraft, since the structures of an aircraft includes many natural corners which would double-bounce the signal. The addition of reflectors with dielectric lenses (radar signal enhancement devices) top and bottom would provide for positive, strong signal return.

These aircraft reflectors could be made to oscillate at standard rates for positive identification of and distinction between aircraft with the same traffic pattern.

The technique uses existing radar systems, serving as an adjunct when rain clutter would otherwise make them unreliable.

For navigation, simple aluminum reflectors in an inexpensive plastic covering would be placed in patterns which would spell out the desired airway information to the radar observer. Similar reflectors would mark mountains, tall buildings and towers, and other navigational hazards. These reflectors would require no maintenance once placed, and the unit cost would be minimal, so that large numbers of them could be used, and their reliability could be extremely high.

Navy and Coast Guard programs exist at present to evaluate the double bounce circular polarization technique as an all-weather sea navigation device.

New System for Automatic Storage and Retrieval of Information.—A new machine which can automatically search a 32,000-page file and present desired information in less time than it takes to drink a cup of coffee has been developed by FMA, Inc., of El Segundo, Calif. Called FileSearch, the machine combines new advances in optical and electronic techniques and is able to retrieve stored information from micro-filmed files at the fastest rate yet known.

Because the system stores 32,000 standard-sized magazine pages on a single reel of microfilm, FileSearch can accommodate over 1,600,000 pages of information—enough to crowd 40 four-drawer file cabinets—in a single file cabinet. When information is desired the machine automatically searches the microfilm reels at the rate of 6400 pages a minute, making copies of the requested material.

With FileSearch, a researcher can locate in a few minutes all the data on a specific subject from an accumulation of thousands of pages of reports or documents. In an average working day he can obtain selected data from over three million pages on file.

George S. Fenn, FMA President, said FileSearch has been designed to meet the commercial need for an economic and practical system of information storage and retrieval.

The FMA FileSearch system is basically composed of a recording unit and a retrieval unit. The recording unit photographs files of documents along with a description of each document's contents which is coded in the form of opaque spots. These are stored together on reels of microfilm.

Unlike any other information handling system, FileSearch—from its request card input to its printed hard copy or film output—is integrated into a single console (71 × 55 × 50 in.) except for a recording camera and simple indexing machine. No associated units such as computers, card readers, etc. are required.

The FileSearch retrieval unit is that part of the system which searches for and finds the facts desired. It can check 6400 standard pages ($8\frac{1}{2} \times 11$) on film per minute and select out for viewing or printing—hard copy or film—any of those pages containing the type of information requested. Requests for information originally are fed into the machine by punched cards. The system can handle up to six requests simultaneously.

The facts requested from the masses of information stored on film are instantaneously projected on the FileSearch screen for viewing by the requestor. If permanent, usable records on paper are required, the machine produces an immediate hard copy print of the document.

FMA, Inc. has put major emphasis on compactness and simplicity of operation in designing the FileSearch system. The system can be installed in the average office and operated by the usual office personnel, and its flexibility in indexing makes it compatible with existing office systems.

"Today, virtually all forms of enterprise find themselves being stifled by the avalanches of data," observed Mr. Fenn. "Activities in many fields are slowed down and made more costly by the problem of gaining access to vital information. Electronics systems offer the only solutions to indexing, storing and retrieving information at the speeds necessary to keep ahead of today's demands."

One of the most acute problem areas, Mr. Fenn said, is in the all-out program in science. Engineers and scientists are facing a flood of data—some 50,000 to 100,000 technical journals are published in more than

60 languages and new journals are appearing at the rate of at least two a day.

This adds up to between one and two million scientific articles and reports per year. To avoid duplicative research and to know what is being accomplished in special fields, scientists must be aware of this data.

Also, researchers pointed out, the nation's population growth has increased records of all kinds both in quantity and in scope to create the same problems for business and government. Greatest usefulness of File-Search is in libraries, industrial firms, military organizations, government departments and similar activities which are struggling with the twin problems of mountains of data and the heavy costs of maintaining bulky files and personnel to dig out information requested.

Information handling methods used for the past 500 years are lagging behind the modern deluge of data. It has been estimated by economists that not finding information stored is costing the United States more than a billion dollars a year, Mr. Fenn said.

F-100F Simulates Boost Glide Vehicle.—An F-100F test aircraft is cutting a scientific swath around the Aeronautical Systems Division so that future aerospace vehicles may someday land on conventional runways.

Normally, an F-100F and other present jet aircraft approach a landing flat—almost parallel with the earth—and with power on. Advanced vehicles, with fuel limitations and lacking the lifting power of modern aircraft, will make steeper approaches to land with no power.

With the alterations made in ASD shops for the Flight and Engineering Test Group, the modified F-100F is able to simulate the landing approach of boost glide vehicles. What is learned from research with this aircraft may eliminate the need for long runways for landing vehicles like Dyna-Soar and the X-15.

Because of these alterations, the modified F-100F Super Sabre takes longer to become airborne and once aloft it flies more slowly than its sister aircraft. The modified plane has a maximum speed of 460 miles per hour, while standard F-100's have flown 810 miles per hour.

The test aircraft makes a steep angle approach and lands at approximately 230 miles per hour. Standard F-100F's land at approximately 155 miles per hour.

First step in modifying the aircraft at ASD was to remove the afterburner and drag chute and install a thrust reverser. A speed brake three times normal size then was built for the aircraft and variable wing flaps with boundary layer control were installed.

Other modifications to the F-100F included installation of a landing indicator to give information on height above ground and the speed at which an approach is being made.

When the aircraft is about 6000 ft. above ground, the test pilot operates the thrust reverser. This diverts the thrust of the plane's turbojet engine forward, slowing the plane and aiding the steep landing. Hot gases from the engine are deflected out and forward through "holes" in the thrust reverser at angles of about 30 deg.

The 33-ft. square speed brake is lowered, providing about two-thirds more drag than the regular speed brake. The regular brake is only 12 sq. ft.

As the plane lands, air is blown over the wing flaps to increase lift. The combination of this and the thrust reverser produces the lift-drag conditions of advanced aerospace vehicles.

Tests are now being conducted in the modified Super Sabre to determine whether manual control or autopilot will be best for steep landings.

Radioactive Wastes Fixed in Glass.—With the expansion of the atomic energy industry, much attention is being devoted all over the world to the development of effective, safe and inexpensive methods for the disposal of radioactive wastes.

Scientists and engineers of the Chemical Engineering Division, Atomic Energy Research Establishment of the U.K. Atomic Energy Authority at Harwell, are developing a method which is expected to simplify the storage of highly radioactive waste.

Laboratory work in the Chemistry and Chemical Engineering Divisions over the last few years has shown that it may be possible to store these materials by converting them into insoluble glass-like solids. Design of a pilot plant to handle 1000 curies of active waste per batch has now reached an advanced stage. The new unit will provide data on which the design of a full-scale plant might be based.

The glass is produced by making a slurry of silica and borax in a nitric acid solution of the concentrated liquid wastes, which, when evaporated to red heat, sinters and melts, and on cooling solidifies to a glass of chosen composition. A typical product might contain from 20% to 30%, by weight, of waste oxides.

Experimental work has shown the properties of the glass remain substantially unchanged when samples are subjected to a radiation dose of 10^{11} rads. This dose is comparable with the dose received by a glass incorporating high activity waste which has been stored for 500 years.

Magnetic Device Aids Cleft Palate Patients.—Cleft palate patients with no teeth now will be able satisfactorily to correct their speech defects through a unique magnetic device created in the Cleft Lip and Palate Institute of Northwestern University. The device makes it possible to correct their long-standing problems of unclear nasal speech and difficulties with proper eating and swallowing.

The apparatus was described for the first time by Morton S. Rosen, D.D.S., assistant professor of prosthetic dentistry at Northwestern and principal designer of the appliance. It was demonstrated to dentists gathered for a two-day postgraduate course, "The Prosthetic and Orthodontic Treatment of Cleft Palate Patients," held at Northwestern Dental School.

The device makes it possible to close the cleft palate patient's gaping hole in the roof of the mouth despite the fact that there are no teeth to attach the device to. Previously such patients have been severely handicapped since there was no satisfactory way to close this abnormal opening between the mouth and nose cavity without using unpleasant and inconvenient adhesives.

"The device should prove to be especially beneficial to persons with cleft palates in the advanced age groups who require dentures," Dr. Rosen said.

Besides aiding cleft palate patients who have trouble keeping dentures in, the device will probably be of great benefit and wide application in patients who have had part of the roof of the mouth destroyed by necessary surgery to remove cancer, or by accidental injuries, such as gunshot wounds, damaging the palate.

The apparatus is effective only for patients with a fairly wide cleft, Dr. Rosen emphasized.

For the apparatus, Dr. Rosen and his associates developed a system of magnets which anchor an upper denture to a gum-colored plastic bulb (obturator) which fits into and above the opening of the cleft palate and effectively seals it off from the nasal cavity.

The danger of a patient swallowing the bulb, Dr. Rosen pointed out, is prevented by an ingenious metal hinge —much like a door hinge. This hinge permanently attaches the back end of the prosthetic bulb to the end of the upper denture.

The magnets are self-centering. One magnet is simply embedded on the underside of the bulb, and it centers itself upon the other magnet embedded in the top of the full denture. In effect, the bulb supports and holds the denture in place.

One woman patient—a cleft palate victim at birth—has been wearing the Institute's new appliance for nearly a year and with excellent results, Dr. Rosen reported. For the first time in her life, she is able to enjoy normal chewing and normal speaking habits.

Drilling Pierces Second Layer of Ocean Floor.—The National Academy of Sciences and the National Science Foundation has announced that the experimental drilling ship, *CUSS I*, operating in 12,000 ft. of water off the west coast of Mexico, has obtained, for the first time, a sample of the so-called second layer under the softer sediments of the ocean floor. This layer, whose presence had been previously indicated to scientists only by reflections of seismic soundings, has been revealed as basalt, a hard rock formed by the solidification of molten material within the earth.

The announcement was based upon the following telegram from Willard Bascomb, director of the experimental drilling program for Project Mohole, to Dr. Detlev W. Bronk, president of the Academy:

"On April 1, at 1600, second hole of experimental deep-sea drilling project at Guadalupe Island site passed completely through soft sediments into basalt.

"This hole, which was begun at 1930 on March 31, reached a depth of 490 feet by 0630 the following morning. Coring then began and cores of soft, gray-green clay of Miocene age were obtained to a depth of 560 feet. At that depth the drilling rate abruptly decreased and when the core barrel was retrieved, it was found to contain a hard, fine-grained basalt.

"This is apparently the upper surface of the so-called second layer, which has long puzzled oceanographers, for its depth corresponds to that predicted by seismic surveys.

"The reaching and sampling of these layers was a major scientific objective of the present test and its identification is a milestone in the exploration of the ocean. All aboard *CUSS I* are jubilant.

"A second core of basalt 10 feet long was then obtained. Drilling continues.

"Wind and waves have been high but in a week on station the drilling ship is not believed to have moved over 350 feet from its intended position.

"The weather now appears to be moderating."

In making public the contents of the telegram, Dr. Bronk said: "This is history-making news of further progress in drilling through the crust of the earth by which we shall gain new understanding of the earth's structure and new evidence concerning the history of life.

"That we should in two short days hear that the previously unexceeded

penetration of the ocean floor has been more than doubled is spectacular evidence of the rate at which scientific progress is made in these exciting times."

Radar Altimeters Fooled by Polar Ice and Snow.—Studies of radio communications in the frozen polar regions have had the lifesaving byproduct of yielding new information on the behavior of aircraft radar altimeters. This was disclosed in a paper presented by two U. S. Army Signal Corps engineers at the International Convention of the Institute of Radio Engineers.

Studies of the behavior of radio waves over the very thick ice caps of Antarctica and Greenland showed that at certain frequencies the radio signals "ignore" the ice caps and penetrate to the soil which may be as much as 1000 ft. below the ice.

The two speakers, Amory H. Waite and S. J. Schmidt of the Siganl Corps laboratories at Ft. Monmouth, N. J., told an audience of engineers at New York's Coliseum that planes flying over the poles could have radar altimeters showing safe clearance above the ground, while the aircraft might be actually skimming the ice cap or about to crash into it.

The paper reported the findings of a 5-year study of radar altimeters, summarized their shortcomings and outlined safety precautions for pilots flying polar routes. One of the authors, a veteran of 10 Arctic and 8 Antarctic expeditions pointed out that over one-ninth of the earth's surface is covered by coatings of ice and snow thick enough to affect radar altimeters.

Remote Production Control of Oil Industry.—An electronic system that keeps a constant vigil over widely separated petroleum wells or cross-country pipelines, reporting breakdowns, relaying instructions from a central control point and logging vital production data, has been announced by C. M. Lewis, Manager, RCA Industrial Computer Systems Department.

Mr. Lewis said the Automatic Logging Electronic Reporting and Telemetering system (ALERT), being introduced by RCA, is "virtually foolproof."

The system employs the RCA 130 Industrial Data Transmission Link to relay performance information and control instructions to and from individual wellheads, including pumping stations 50 miles or more off-shore, he noted, adding:

"ALERT enables the operator of a computer to keep in constant touch with all the wellheads under control, making production changes instantaneously or according to a desired program to maintain the daily output required. In the case of the pipeline operator, ALERT provides up-to-the second reports on pressures, pumping station failures or power breakdowns. The system supervisor is as close to his most distant charge as the control panel beside him."

The new RCA system was designed primarily, and is being produced at the Natick, Mass., industrial computer plant of RCA.

Explaining the system, Mr. Lewis said: "Heart of the ALERT system is the central control station. Here, perhaps hundreds of miles from the farthest wellhead or pumping station being covered, the operator supervises a control panel with pushbuttons to issue orders, a display panel showing the petroleum field or pipeline network in graphic form, an alarm printer, a production logger, test data logger, automatic typewriter and paper tape equipment.

"On the control panel, when all is

well, a battery of signal lights glow dimly. When trouble occurs, lights flash an alarm.

"In the event of a generator failure at a remote location, that information is relayed to the control center as a spare generator is activated. The breakdown is duly noted by the ALERT system and a repair crew dispatched to the scene."

Mr. Lewis pointed out that periodic tests for sulfur content and other chemical properties of a well's output can be handled on a pre-programmed basis by the ALERT system. He said that once the program has been set up, tests can be carried out on a number of wells in sequence, including the gathering of information indicating a well's expected life span.

"A wide variety of transmission media can be used with the RCA 130 equipment—very high frequency radio, telephone or telegraph lines, submarine cable or microwave," he continued.

"Because petroleum operators are limited to specific production levels by government regulation, the logging of production data forms a vital service of the ALERT system. The information is recorded by a logging typewriter automatically, or in the form of punched cards or perforated paper tape for use with an electronic computer."

Special Radiation Paint.—Protection from the heat of nuclear blasts will be given Air Force planes by a simple thing—paint. The special paint, technically identified as a polyurethane coating, developed by the Air Research and Development Command (ARDC), will be sprayed on aircraft to protect them from the tremendous heat generated by a nuclear blast or by firing of the air-to-surface missiles with which they may be armed.

It also will be sprayed on air-to-surface missiles themselves to protect them from possible thermal radiation.

Polyurethane coating, the purest white which retains its color, was developed by ARDC's Wright Air Development Division (WADD), Dayton, Ohio. It resists heat, moisture, synthetic lubricants, hydraulic fluid, and thermal radiations.

Standard enamels were used in early applications aimed toward protecting the aircraft, but did not prove satisfactory because they were affected by atmospheric heat and moisture. These enamels also discolor on exposure and pick up soil rapidly, causing a loss in their protective qualities.

The new coating keeps the metal parts of the aircraft cool. Service-tested for more than a year, the coating has resisted erosion in tests conducted under unfavorable environmental conditions.

Electronic Obstacle Detector for the Blind.—Two infrared devices that detect obstacles for the blind were reported by J. Malvern Benjamin, Jr. and T. A. Benham of Biophysical Electronics, Inc., New Hope, Pa., at the International Convention of the Institute of Radio Engineers meeting at the Coliseum.

One device, called an obstacle detector, uses a special infrared lamp to send out invisible pulses of radiation which are reflected back to the instrument by obstacles. A photoelectric receiver picks up the signal and causes a stimulator, located in the handle of the instrument, to vibrate. With training, these vibrations can be interpreted by the blind person to indicate distance from the obstacle. A model has been built and field tested at distances ranging from 2 to 8 ft.

Experimental models have also been made of a curb detector, although

more development work is necessary. Basically, the approach is to provide a steady source of infrared light unaffected by the walking motion. This light is focused on the sidewalk about 5 ft. ahead of the user. As a curb is approached, the beam of light "jumps," and the momentary disappearance of the light from its spot on the sidewalk is transferred by a photosensitive device to a small loudspeaker which emits an audible tone. The curb detector also detects holes, up-steps and down-steps.

The ultimate aim is to incorporate both the curb and obstacle detectors into a single compact unit. When this has been accomplished, electronics will have made an important contribution to the field of travel aids for the blind.

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This is an inch-long single crystal of beryllium.

That it has undergone such severe bending without fracture is evidence of a remarkable degree of ductility, tending to deny the long-held conviction that beryllium is extremely brittle. The behavior pictured here was recently observed in the Laboratories of The Franklin Institute. The specimen was prepared by a floating-zone refining technique, as part of a program sponsored by the Bureau of Naval Weapons. The purpose of this program is to determine whether the generally observed brittleness of beryllium is an intrinsic property of the metal or merely a result of impurities within the metal. Recent tests in our Laboratories tend to support the latter possibility. All specimens of zone melted crystals have exhibited extensive basal plane slip—in contrast to other studies in which fracture has occurred after only very limited basal slip.

This development raises important new questions, which our staff is now trying to answer: Is it possible, for example, to reproduce this ductility in polycrystalline beryllium? What impurities are responsible for the brittle behavior of beryllium? And through what mechanism do impurities contribute to brittleness?

This challenging program is one of the many highly interesting research programs now being carried out for government agencies and industry in our Laboratories. To obtain additional information concerning activities or staff positions in the Laboratories, address *Director of Personnel,* The Franklin Institute.

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